CALIBRATED MANIFOLDS AND GAUGE THEORY

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Abstract. By a theorem of Mclean, the deformation space of an associative submanifold Y of an integrable G_2-manifold (M, φ) can be identified with the kernel of a Dirac operator $\not{D} : \Omega^0(\nu) \to \Omega^0(\nu)$ on the normal bundle ν of Y. Here, we generalize this to the non-integrable case, and also show that the deformation space becomes smooth after perturbing it by natural parameters, which corresponds to moving Y through 'pseudo-associative' submanifolds. Infinitesimally, this corresponds to twisting the Dirac operator $\not{D} \mapsto \not{D}_A$ with connections A of ν. Furthermore, the normal bundles of the associative submanifolds with $Spin^c$ structure have natural complex structures, which helps us to relate their deformations to Seiberg-Witten type equations.

If we consider G_2 manifolds with 2-plane fields (M, φ, Λ) (they always exist) we can split the tangent space TM as a direct sum of an associative 3-plane bundle and a complex 4-plane bundle. This allows us to define (almost) Λ-associative submanifolds of M, whose deformation equations, when perturbed, reduce to Seiberg-Witten equations, hence we can assign local invariants to these submanifolds. Using this we can assign an invariant to (M, φ, Λ). These Seiberg-Witten equations on the submanifolds are restrictions of global equations on M. We also discuss similar results for the Cayley submanifolds of a $Spin(7)$ manifold.

0. Introduction

We first study deformations of associative submanifolds Y^3 of a G_2 manifold (M^7, φ), where $\varphi \in \Omega^3(M)$ is the G_2 structure. We prove a generalized version of the McLean's theorem where integrability condition of the underlying G_2 structure is not necessary. This deformation space might be singular, but by perturbing it with some natural parameters it can be made smooth. This amounts to deforming Y through the associatives in (M, φ) with varying φ, or alternatively deforming Y through the pseudo-associative submanifolds (Y's whose tangent planes become associative after rotating by a generic element of the gauge group of TM). Infinitesimally, these perturbed deformations correspond to the kernel of the twisted Dirac operator $\not{D}_A : \Omega^0(\nu) \to \Omega^0(\nu)$, twisted by some connection A in $\nu(Y)$.

Date: February 1, 2008.

1991 *Mathematics Subject Classification.* 53C38, 53C29, 57R57.

Key words and phrases. deformation of calibrated manifolds.

First named author is partially supported by NSF grant DMS 9971440.

The associative submanifolds with $Spin^c$ structures in (M,φ) are useful objects to study, because their normal bundles have natural complex structures. Also we can view (M,φ) as an analog of a symplectic manifold, and view a non-vanishing 2-plane field Λ on M as an analog of a complex structure taming φ. Note that 2-plane fields are stronger versions of $Spin^c$ structures on M^7, and they always exist by [T]. The data (M^7,φ,Λ) determines an interesting splitting of the tangent bundle $TM = \mathbf{E}\oplus\mathbf{V}$, where $\mathbf{E}$ is the bundle of associative 3-planes, and $\mathbf{V}$ is the complementary 4-plane bundle with a complex structure, which is a spinor bundle of $\mathbf{E}$. Then the integral submanifolds Y^3 of $\mathbf{E}$, which we call Λ-*associative submanifolds*, can be viewed as analogues of J-holomorphic curves; because their normal bundles come with an almost complex structure. Even if they may not always exist, their perturbed versions, i.e. *almost* Λ-*associative submanifolds*, always do. Almost Λ-associative submanifolds are the transverse sections of the bundle $\mathbf{V}\to M$. We can deform such Y by using the connections in the determinant line bundle of $\nu(Y)$ and get a smooth deformation space, which is described by the twisted Dirac equation. Then by constraining this new variable with another natural equation we arrive to Seiberg-Witten type equations for Y. So we can assign an integer to Y, which is invariant under small isotopies through almost Λ-assocative submanifolds.

In fact it turns out that (M^7,φ,Λ) gives a finer splitting $TM=\bar{\mathbf{E}}\oplus\xi$, where $\bar{\mathbf{E}}$ is a 6-plane bundle with a complex structure, and ξ is a real line bundle. In a way this structure of (M,φ) mimics the structure of (Calabi-Yau)$\times S^1$ manifolds, and by 'rotating' ξ inside of TM we get a new insight for so-called "Mirror manifolds" which is investigated in [AS1].

There is a similar process for the deformations of Cayley submanifolds $X^4\subset N^8$ of a Spin(7) manifold (N^8,Ψ), which we discuss at the end. So in a way Λ-associative (or Cayley) manifolds in a G_2 (or $Spin(7)$) manifold, behave much like higher dimensional analogue of holomorphic curves in a Calabi-Yau manifold.

We would like to thank MSRI, IAS, Princeton and Harvard Universities for providing a stimulating environment where this paper is written, and we thank R. Kirby and G. Tian for continuous encouragement. The first named author thanks to R. Bryant and C. Taubes for stimulating discussions and useful suggestions.

1. Preliminaries

Here we first review basic properties of the manifolds with special holonomy (most material can be found in [B2], [B3], [H], [HL]), and then proceed to prove some new results. Recall that the set of octonions $\mathbb{O} = \mathbb{H} \oplus l\mathbb{H} = \mathbb{R}^8$ is an 8-dimensional division algebra generated by $< 1, i, j, k, l, li, lj, lk >$. On the set of the imaginary octonions $im\mathbb{O} = \mathbb{R}^7$ we have the cross product operation $\times : \mathbb{R}^7 \times \mathbb{R}^7 \to \mathbb{R}^7$, defined by $u \times v = im(\bar{v}.u)$. The exceptional Lie group G_2 can be defined as the linear automorphisms of $im\mathbb{O}$ preserving this cross product operation, $G_2 = Aut(\mathbb{R}^7, \times)$. There is also another useful description in terms of the orthogonal 3-frames in $\mathbb{R}^7$:

$$G_2 = \{(u_1, u_2, u_3) \in (im\mathbb{O})^3 \mid\ < u_i, u_j >= \delta_{ij},\ < u_1 \times u_2, u_3 >= 0\ \} \tag{1}$$

Alternatively, G_2 can be defined as the subgroup of the linear group $GL(7, \mathbb{R})$ which fixes a particular 3-form $\varphi_0 \in \Omega^3(\mathbb{R}^7)$. Denote $e^{ijk} = dx^i \wedge dx^j \wedge dx^k \in \Omega^3(\mathbb{R}^7)$, then

$$G_2 = \{A \in GL(7, \mathbb{R}) \mid A^*\varphi_0 = \varphi_0\ \}$$

$$\varphi_0 = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356} \tag{2}$$

Definition 1. *A smooth 7-manifold M^7 has a G_2 structure if its tangent frame bundle reduces to a G_2 bundle. Equivalently, M^7 has a G_2 structure if there is a 3-form $\varphi \in \Omega^3(M)$ such that at each $x \in M$ the pair $(T_x(M), \varphi(x))$ is isomorphic to $(T_0(\mathbb{R}^7), \varphi_0)$.*

Here are some useful properties, discussed more fully in [B2]: Any G_2 structure φ on M^7 gives an orientation $\mu \in \Omega^7(M)$ on M, and this μ determines a metric $g = \langle\ ,\ \rangle$ on M, and a cross product structure $\times$ on its tangent bundle of M as follows: Let i_v denote the interior product with a vector v then

$$\langle u, v\rangle = [i_u(\varphi) \wedge i_v(\varphi) \wedge \varphi]/6\mu \tag{3}$$

$$\varphi(u, v, w) = \langle u \times v, w\rangle \tag{4}$$

To emphasize the dependency on φ sometimes g is denoted by g_φ. In particular, the 14-dimensional Lie group G_2 imbeds into $SO(7)$ subgroup of $GL(7, \mathbb{R})$. Note that because of the way we defined $G_2 = G_2^{\varphi_0}$, this imbedding is determined by φ_0.

Since $GL(7, \mathbb{R})$ acts on $\Lambda^3(\mathbb{R}^7)$ with stabilizer G_2, its orbit $\Lambda^3_+(\mathbb{R}^7)$ is open for dimension reasons, so the choice of φ_0 in the above definition is generic (in fact it has two orbits containing $\pm\varphi_0$). G_2 has many copies G_2^φ inside $GL(7, \mathbb{R})$, which are all conjugate to each other, since G_2 has only one 7 dimensional representation. Hence the space of G_2 structures on M^7 are identified with the sections of the bundle:

$$\mathbb{RP}^7 \simeq GL(7, \mathbb{R})/G_2 \to \Lambda^3_+(M) \longrightarrow M \tag{5}$$

which are called *the positive* 3*-forms*, these are the set of 3-forms $\Omega^3_+(M)$ that can be identified pointwise by φ_0. Each G_2^{φ} imbeds into a conjugate of one standard copy $SO(7) \subset GL(7,\mathbb{R})$. The space of G_2 structures φ on M, which induce the same metric on M, that is all φ's for which the corresponding G_2^{φ} lies in the standard $SO(7)$, are the sections of the bundle (whose fiber is the orbit of φ_0 under $SO(7)$):

$$\mathbb{RP}^7 = SO(7)/G_2 \to \tilde{\Lambda}^3_+(M) \longrightarrow M \tag{6}$$

which we will denote by $\tilde{\Omega}^3_+(M)$. The set of smooth 7-manifolds with G_2-structures coincides with the set of 7-manifolds with spin structure, though this correspondence is not $1-1$. This is because $Spin(7)$ acts on S^7 with stabilizer G_2 inducing the fibrations

$$G_2 \to Spin(7) \to S^7 \to BG_2 \to BSpin(7)$$

and so there is no obstruction to lifting maps $M^7 \to BSpin(7)$ to BG_2, and there are many liftings. Cotangent frame bundle $\mathcal{P}^*(M) \to M$ of a manifold with G_2 structure (M,φ) can be expressed as $\mathcal{P}^*(M) = \cup_{x\in M}\, \mathcal{P}^*_x(M)$, where each fiber is:

$$\mathcal{P}^*_x(M) = \{u \in Hom(T_x(M),\mathbb{R}^7) \mid u^*(\varphi_0) = \varphi(x)\ \}$$

Throughout this paper we will denote the cotangent frame bundle by $\mathcal{P}^*(M) \to M$ and its adapted frame bundle by $\mathcal{P}(M)$. They can be G_2 or $SO(7)$ frame bundles; to emphasize it sometimes we will specify them by the notations $\mathcal{P}_{SO(7)}(M)$ or $\mathcal{P}_{G_2}(M)$. Also we will denote the sections of a bundle $\xi \to Y$ by $\Omega^0(Y,\xi)$ or simply by $\Omega^0(\xi)$, and the bundle valued p-forms by $\Omega^p(\xi) = \Omega^0(\Lambda^p T^*Y \otimes \xi)$, and the *sphere bundle* of ξ by $S(\xi)$. There is a notion of a G_2 structure φ on M^7 being *integrable*, which corresponds to φ being an harmonic form:

Definition 2. *A manifold with G_2 structure (M,φ) is called a G_2 manifold if the holonomy group of the Levi-Civita connection (of the metric g_φ) lies inside of G_2. Equivalently (M,φ) is a G_2 manifold if φ is parallel with respect to the metric g_φ i.e. $\nabla_{g_\varphi}(\varphi) = 0$; this condition is equivalent to $d\varphi = 0 =\ d(*_{g_\varphi}\varphi)$.*

In short one can define a G_2 manifold to be any Riemannian manifold (M^7,g) whose holonomy group is contained in G_2, then φ and the cross product $\times$ come as a consequence. It turns out that the condition φ being harmonic is equivalent to the condition that at each point $x_0 \in M$ there is a chart $(U,x_0) \to (\mathbb{R}^7,0)$ on which φ equals to φ_0 up to second order term, i.e. on the image of U

$$\varphi(x) = \varphi_0 + O(|x|^2) \tag{7}$$

Remark 1. *For example if (X^6,ω,Ω) is a complex 3-dimensional Calabi-Yau manifold with Kähler form ω, and a nowhere vanishing holomorphic 3-form Ω, then $X \times S^1$ has holonomy group $SU(3) \subset G_2$, hence is a G_2 manifold. In this case*

$$\varphi = Re\,\Omega + \omega \wedge dt. \tag{8}$$

Definition 3. *Let (M,φ) be a manifold with a G_2 structure. A 4-dimensional submanifold $X \subset M$ is called an* co-associative *if $\varphi|_X = 0$. A 3-dimensional submanifold $Y \subset M$ is called an* associative *if $\varphi|_Y \equiv vol(Y)$; this condition is equivalent to $\chi|_Y \equiv 0$, where $\chi \in \Omega^3(M,TM)$ is the tangent bundle valued 3-form defined by the identity:*

$$\langle \chi(u,v,w), z\rangle = *\varphi(u,v,w,z) \tag{9}$$

The equivalence of these conditions follows from the 'associator equality' of [HL]

$$\varphi(u,v,w)^2 + |\chi(u,v,w)|^2/4 = |u\wedge v\wedge w|^2 \tag{10}$$

In general, if $\{e^1,e^2,..,e^7\}$ is any orthonormal coframe on (M,φ), then the expression (2) for φ hold on a chart. By calculation $*\varphi$, and using (9) we can calculate the expression of χ (note the error in the the second term of 6th line of the corresponding formula (5.4) of [M]):

$$*\,\varphi = e^{4567} + e^{2367} + e^{2345} + e^{1357} - e^{1346} - e^{1256} - e^{1247} \tag{11}$$

$$\begin{aligned}
\chi = \quad & (e^{256} + e^{247} + e^{346} - e^{357})\ e_1 \\
& + (-e^{156} - e^{147} - e^{345} - e^{367})\ e_2 \\
& + (e^{245} + e^{267} - e^{146} + e^{157})\ e_3 \\
& + (-e^{567} + e^{127} + e^{136} - e^{235})\ e_4 \\
& + (e^{126} + e^{467} - e^{137} + e^{234})\ e_5 \\
& + (-e^{457} - e^{125} - e^{134} - e^{237})\ e_6 \\
& + (e^{135} - e^{124} + e^{456} + e^{236})\ e_7
\end{aligned}$$

Also χ can be expressed in terms of cross product operation (c.f. [H], [HL], [K]):

$$\chi(u,v,w) = -u\times(v\times w) - \langle u,v\rangle w + \langle u,w\rangle v \tag{12}$$

When $d\varphi = 0$, the associative submanifolds are volume minimizing submanifolds of M (calibrated by φ). Even in the general case of a manifold with a G_2 structure (M,φ), the form χ imposes an interesting structure near associative submanifolds:

Notice (9) implies that, χ maps every oriented 3-plane in $T_x(M)$ to the orthogonal subspace $T_x(M)^{\perp}$, so if we choose local coordinates $(x_1,...,x_7)$ for M^7 we get

$$\chi = \sum a_J^{\alpha}\, dx^J \otimes \frac{\partial}{\partial x_\alpha} \tag{13}$$

where $dx^J = dx^i\wedge dx^j\wedge dx^k$, and the summation is taken over the multi-index $J=\{i,j,k\}$ and α such that $\alpha\notin J$. So if $Y\subset M$ is given by (x_1,x_2,x_3) coordinates, then locally the condition Y to be associative is given by the equations:

$$a_{123}^{\alpha} = 0 \tag{14}$$

From (9) it is easy to calculate $a_{ijk}^{\alpha} = *\varphi_{ijks}g^{s\alpha}$, where $g^{-1} = (g^{ij})$ is the inverse of the metric $g = (g_{ij})$, and of course the metric g can be expressed in terms of φ. By evaluating χ on the orientation form of Y we get a normal vector field so:

Lemma 1. *To any* 3*-dimensional submanifold* $Y^3 \subset (M,\varphi)$*,* χ *associates a normal vector field, which vanishes when* Y *is associative.*

Hence χ defines an interesting flow on 3 dimensional submanifolds of (M,φ), fixing associative submanifolds. On the associative submanifolds with a $Spin^c$ structure, χ rotates their normal bundles and imposes a complex structure on them:

Lemma 2. *To any associative manifold* $Y^3 \subset (M,\varphi)$ *with a non-vanishing oriented* 2*-plane field,* χ *defines an almost complex structure on its normal bundle* $\nu(Y)$ *(notice that in particular any coassociative submanifold* $X \subset M$ *has an almost complex structure if its normal bundle has a non-vanishing section).*

Proof. Let $L \subset \mathbb{R}^7$ be an associative 3-plane, that is $\varphi|_L = vol(L)$. Then to every pair of orthonormal vectors $\{u,v\} \subset L$, the form χ defines a complex structure on the orthogonal 4-plane $L^{\perp}$, as follows: Define $j : L^{\perp} \to L^{\perp}$ by

$$j(X) = \chi(u,v,X) \tag{15}$$

This is well defined i.e. $j(X) \in L^{\perp}$, because when $w \in L$ we have:

$$<\chi(u,v,X),w>= *\varphi(u,v,X,w) = -*\varphi(u,v,w,X) =<\chi(u,v,w),X>= 0$$

Also $j^2(X) = j(\chi(u,v,X)) = \chi(u,v,\chi(u,v,X)) = -X$. We can check the last equality by taking an orthonormal basis $\{X_j\} \subset L^{\perp}$ and calculating

$$\begin{aligned} <\chi(u,v,\chi(u,v,X_i)),X_j> &= *\varphi(u,v,\chi(u,v,X_i),X_j) = \\ -*\varphi(u,v,X_j,\chi(u,v,X_i)) &= -<\chi(u,v,X_j),\chi(u,v,X_i)>= -\delta_{ij} \end{aligned}$$

The last equality holds since the map j is orthogonal, and the orthogonality can be seen by polarizing the associator equality (10), and by noticing $\varphi(u,v,X_i) = 0$. Observe that the map j only depends on the oriented 2-plane $l =<u,v>$ generated by $\{u,v\}$. So the result follows. □

In fact, for any unit vector field ξ on an associative Y (i.e. a $Spin^c$ structure) defines a complex structure $J_{\xi} : \nu(Y) \to \nu(Y)$ by $J_{\xi}(z) = z \times \xi$, and the complex structure defined in Lemma 2 corresponds to $J_{u\times v}$, because from (12):

$$\chi(u,v,z) = \chi(z,u,v) = -z\times(u\times v) - \langle z,u\rangle v + \langle z,v\rangle u = J_{v\times u}(z).$$

Also recall that the complex structures on any $SO(4)$ bundle such as $\nu \to Y$ are given by the unit sections of the associated $SO(3)$ bundle $\lambda_{+}(\nu) \to Y$, which is induced by the left reductions $SO(4) = (SU(2)\times SU(2))/\mathbb{Z}_2 \to SU(2)/\mathbb{Z}_2 = SO(3)$.

Definition 4. *A Riemannian 8-manifold* (N^8, g) *is called a* Spin(7) manifold *if the holonomy group of its Levi-Civita connection lies in* $Spin(7) \subset GL(8, \mathbb{R})$.

Equivalently a $Spin(7)$ manifold (N, Ψ) is a Riemannian 8-manifold with a triple cross product $\times$ on its tangent bundle, and a harmonic 4-form $\Psi \in \Omega^4(N)$ with

$$\Psi(u, v, w, z) = g(u \times v \times w, z)$$

It is easily checked that if (M, φ) is a G_2 manifold, then $(M \times S^1, \Psi)$ is a $Spin(7)$ manifold where $\Psi = \varphi \wedge dt - *\varphi$.

Definition 5. *A 4-dimensional submanifold* X *of a* $Spin(7)$ *manifold* (N, Ψ) *is called* Cayley *if* $\Psi|_X \equiv vol(X)$. *This is equivalent to* $\tau|_X \equiv 0$ *where* $\tau \in \Omega^4(N, E)$ *is a certain vector-bundle valued 4-form defined by the "four-fold cross product" of the imaginary octonions* $\tau(v_1, v_2, v_3, v_4) = v_1 \times v_2 \times v_3 \times v_4$ *(see* [M], [HL]*).*

2. Grassmann Bundles

Let $G(3, 7)$ be the Grassmann manifold of oriented 3-planes in $\mathbb{R}^7$. Let M^7 be an oriented smooth 7-manifold, and let $\tilde{M} \to M$ be the bundle oriented 3-planes in TM, which is defined by the identification $[p, L] = [pg, g^{-1}L] \in \tilde{M}$:

$$\tilde{M} = \mathcal{P}_{SO(7)}(M) \times_{SO(7)} G(3, 7) \to M. \tag{16}$$

This is just the bundle $\tilde{M} = \mathcal{P}_{SO(7)}(M)/SO(3) \times SO(4) \to \mathcal{P}_{SO(7)}(M)/SO(7) = M$. Let $\xi \to G(3, 7)$ be the universal $\mathbb{R}^3$ bundle, and $\nu = \xi^{\perp} \to G(3, 7)$ be the dual $\mathbb{R}^4$ bundle. Therefore, $Hom(\xi, \nu) = \xi^* \otimes \nu \longrightarrow G(3, 7)$ is the tangent bundle $TG(3, 7)$. ξ, ν extend fiberwise to give bundles $\Xi \to \tilde{M}$, $\mathbb{V} \to \tilde{M}$ respectively, and let Ξ^* be the dual of Ξ. Notice that $Hom(\Xi, \mathbb{V}) = \Xi^* \otimes \mathbb{V} \to \tilde{M}$ is the bundle of vertical vectors $T^v(\tilde{M})$ of $T(\tilde{M}) \to M$, i.e. the tangents to the fibers of $\pi : \tilde{M} \to M$, hence

$$T\tilde{M} \cong T^v(\tilde{M}) \oplus \pi^* TM = (\Xi^* \otimes \mathbb{V}) \oplus \Xi \oplus \mathbb{V}. \tag{17}$$

That is, $T\tilde{M}$ is the vector bundle associated to principal $SO(3) \times SO(4)$ bundle $\mathcal{P}_{SO(7)} \to \tilde{M}$ by the obvious representation of $SO(3) \times SO(4)$ to $(\mathbb{R}^3)^* \otimes \mathbb{R}^4 + \mathbb{R}^3 + \mathbb{R}^4$. The identification (17) is defined up to gauge automorphisms of bundles Ξ and $\mathbb{V}$.

Note that the bundle $\mathbb{V} = \Xi^{\perp}$ depends on the metric, and hence it depends on φ when metric is induced from a G_2 structure (M, φ). To emphasize this fact we can denote it by $\mathbb{V}_{\varphi} \to \tilde{M}$. But when we are considering G_2 structures coming from G_2 subgroups of a fixed copy of $SO(7) \subset GL(7, \mathbb{R})$, they induce the same metric and so this distinction is not necessary.

Let $\mathcal{P}(\mathbb{V}) \to \tilde{M}$ be the $SO(4)$ frame bundle of the vector bundle $\mathbb{V}$, identify $\mathbb{R}^4$ with the quaternions $\mathbb{H}$, and identify $SU(2)$ with the unit quaternions $Sp(1) = S^3$. Recall that $SO(4)$ is the equivalence classes of pairs $[\,q, \lambda\,]$ of unit quaternions

$$SO(4) = (SU(2) \times SU(2))/\mathbb{Z}_2$$

Hence $\mathbb{V} \to \tilde{M}$ is the associated vector bundle to $\mathcal{P}(\mathbb{V})$ via the $SO(4)$ representation

$$x \mapsto qx\lambda^{-1} \tag{18}$$

There is a pair of $\mathbb{R}^3 = im(\mathbb{H})$ bundles over $\tilde{M}$ corresponding to the left and right $SO(3)$ reductions of $SO(4)$, which are given by the $SO(3)$ representations

$$\begin{array}{rl} \lambda_+(\mathbb{V}) : & x \mapsto qx\,q^{-1} \\ \lambda_-(\mathbb{V}) : & y \mapsto \lambda y \lambda^{-1} \end{array} \tag{19}$$

The map $x \otimes y \mapsto xy$ gives actions $\lambda_+(\mathbb{V}) \otimes \mathbb{V} \to \mathbb{V}$ and $\mathbb{V} \otimes \lambda_-(\mathbb{V}) \to \mathbb{V}$; by combining we can think of them as one conjugation action

$$(\lambda_+(\mathbb{V}) \otimes \lambda_-(\mathbb{V})) \otimes \mathbb{V} \to \mathbb{V} \tag{20}$$

If the $SO(4)$ bundle $\mathcal{P}(\mathbb{V}) \to \tilde{M}$ lifts to a $Spin(4) = SU(2) \times SU(2)$ bundle (locally it does), we get two additional bundles over $\tilde{M}$

$$\begin{array}{rl} \mathcal{S} : & y \mapsto qy \\ \mathbb{E} : & y \mapsto y\lambda^{-1} \end{array} \tag{21}$$

They identify $\mathbb{V}$ as a tensor product of two quaternionic line bundles $\mathbb{V} = \mathcal{S} \otimes_{\mathbb{H}} \mathbb{E}$. In particular, $\lambda_+(\mathbb{V}) = ad(\mathcal{S})$ and $\lambda_-(\mathbb{V}) = ad(\mathbb{E})$, i.e. they are the $SO(3)$ reductions of the $SU(2)$ bundles $\mathcal{S}$ and $\mathbb{E}$. Also there is a multiplication map $\mathcal{S} \otimes \mathbb{E} \to \mathbb{V}$. Recall the identifications: $\Lambda^2(\mathbb{V}) = \Lambda^2_+(\mathbb{V}) \oplus \Lambda^2_-(\mathbb{V}) = \lambda_-(\mathbb{V}) \oplus \lambda_+(\mathbb{V}) = \lambda(\mathbb{V}) = gl(\mathbb{V}) = ad(\mathbb{V})$.

2.1. **Associative Grassmann Bundles.**

Now consider the *Grassmannian of associative* 3*-planes* $G^{\varphi}(3,7)$ in $\mathbb{R}^7$, consisting of elements $L \in G(3,7)$ with the property $\varphi_0|_L = vol(L)$ (or equivalently $\chi_0|_L = 0$). G_2 acts on $G^{\varphi}(3,7)$ transitively with the stabilizer $SO(4)$, so it gives the identification $G^{\varphi}(3,7) = G_2/SO(4)$. If we identify the imaginary octonions by $\mathbb{R}^7 = \mathrm{Im}(\mathbb{O}) \cong im(\mathbb{H}) \oplus \mathbb{H}$, then the action of the subgroup $SO(4) \subset G_2$ on $\mathbb{R}^7$ is

$$\begin{pmatrix} \rho(A) & 0 \\ 0 & A \end{pmatrix} \tag{22}$$

where $\rho : SO(4) = (SU(2) \times SU(2))/\mathbb{Z}_2 \to SO(3)$ is the projection of the first factor ([HL]), that is for $[q, \lambda] \in SO(4)$ the action is given by $(x, y) \mapsto (qxq^{-1}, qy\lambda^{-1})$. So the action of $SO(4)$ on the 3-plane $L = im(\mathbb{H})$ is determined by its action on $L^{\perp}$. Now let M^7 be a G_2 manifold. Similar to the construction before, we can construct the bundle of associative Grassmannians over M (which is a submanifold of $\tilde{M}$):

$$\tilde{M}_{\varphi} = \mathcal{P}_{G_2}(M) \times_{G_2} G^{\varphi}(3,7) \to M \tag{23}$$

which is just the quotient bundle $\tilde{M}_\varphi = \mathcal{P}_{G_2}(M)/SO(4) \longrightarrow \mathcal{P}_{G_2}(M)/G_2 = M$. As in the previous section, the restriction of the universal bundles $\xi,\ \nu = \xi^\perp \to G^\varphi(3,7)$ induce 3 and 4 plane bundles $\Xi \to \tilde{M}_\varphi$ and $\mathbb{V} \to \tilde{M}_\varphi$ (by restricting from $\tilde{M}$). Also

$$T\tilde{M}_\varphi \cong T^v(\tilde{M}_\varphi) \oplus \Xi \oplus \mathbb{V} \tag{24}$$

From (22) we see that in the associative case, we have an important identification: $\Xi = \lambda_+(\mathbb{V})$ (as bundles over $\tilde{M}_\varphi$), and the dual of the action $\lambda_+(\mathbb{V}) \otimes \mathbb{V} \to \mathbb{V}$ gives a Clifford multiplication:

$$\Xi^* \otimes \mathbb{V} \to \mathbb{V} \tag{25}$$

In fact this is just the map induced from the cross product operation [AS2]. Recall that $T^v(\tilde{M}) = \Xi^* \otimes \mathbb{V} \to \tilde{M}$ is the subbundle of vertical vectors of $T(\tilde{M}) \to M$. The total space $E(\nu_\varphi)$ of the normal bundle of the imbedding $\tilde{M}_\varphi \subset \tilde{M}$ should be thought of an open tubular neighborhood of $\tilde{M}_\varphi$ in $\tilde{M}$, and it has a nice description:

Lemma 3. *(*[M]*) Normal bundle ν_φ of $\tilde{M}_\varphi \subset \tilde{M}$ is isomorphic to $\mathbb{V}$, and the bundle of vertical vectors $T^v(\tilde{M}_\varphi)$ is the kernel of the Clifford multiplication $c : \Xi^* \otimes \mathbb{V} \to \mathbb{V}$. We have $T^v(\tilde{M})|_{\tilde{M}_\varphi} = T^v(\tilde{M}_\varphi) \oplus \nu_\varphi$, and the following exact sequence over $\tilde{M}_\varphi$*

$$T^v(\tilde{M}_\varphi) \to \Xi^* \otimes \mathbb{V}|_{\tilde{M}_\varphi} \xrightarrow{c} \mathbb{V}|_{\tilde{M}_\varphi} \to 0$$

Hence the quotient bundle, $T^v(\tilde{M})/T^v(\tilde{M}_\varphi)$ is isomorphic to $\mathbb{V}$.

Proof. This is because the Lie algebra inclusion $g_2 \subset so(7)$ is given by

$$\begin{pmatrix} a & \beta \\ -\beta^t & \rho(a) \end{pmatrix}$$

where $a \in so(4)$ is $y \mapsto qy - y\lambda$, and $\rho(a) \in so(3)$ is $x \mapsto qx - xq$. So the tangent space inclusion of $G_2/SO(4) \subset SO(7)/SO(4) \times SO(3)$ is given by the matrix $\beta \in (im\mathbb{H})^* \otimes \mathbb{H}$. Therefore, if we write β as column vectors of three queternions $\beta = (\beta_1, \beta_2, \beta_3) = i^* \otimes \beta_1 + j^* \otimes \beta_2 + k^* \otimes \beta_3$, then $\beta_1 i + \beta_2 j + \beta_3 k = 0$ ([M], [Mc]). $\square$

The reader can consult Lemma 5 of [AS2] for a more self contained proof of this fact, where the Clifford multiplication is identified with the cross product operation.

3. Associative Submanifolds

Any imbedding of a 3-manifold $f : Y^3 \hookrightarrow M^7$ induces an imbedding $\tilde{f} : Y \hookrightarrow \tilde{M}$:

$$\begin{array}{ccc} & & \tilde{M} \supset \tilde{M}_\varphi \\ & \tilde{f} \nearrow & \downarrow \\ Y & \xrightarrow{f} & M \end{array} \tag{26}$$

and the pull-backs $\tilde{f}^*\Xi = T(Y)$ and $\tilde{f}^*\mathbb{V} = \nu(Y)$ give the tangent and normal bundles of Y. Furthermore, if f is an imbedding of an associative submanifold into a G_2 manifold (M, φ), then the image of $\tilde{f}$ lands in $\tilde{M}_\varphi$. We will denote this canonical lifting of any 3-manifold $Y \subset M$ by $\tilde{Y} \subset \tilde{M}$. Also since we have the dependency $\mathbb{V} = \mathbb{V}_\varphi$, we can denote $\nu(Y) = \nu(Y)_\varphi = \nu_\varphi$ when needed.

$\tilde{M}_\varphi$ can be thought of as a universal space parameterizing associative submanifolds of M. In particular, if $\tilde{f} : Y \hookrightarrow \tilde{M}_\varphi$ is the lifting of an associative submanifold, by pulling back we see that the principal $SO(4)$ bundle $\mathcal{P}(\mathbb{V}) \to \tilde{M}_\varphi$ induces an $SO(4)$-bundle $\mathcal{P}(Y) \to Y$, and gives the following vector bundles via the representations:

$$\begin{array}{rcl} \nu(Y) & : & y \mapsto qy\lambda^{-1} \\ T(Y) & : & x \mapsto qx\, q^{-1} \end{array} \tag{27}$$

where $[q, \lambda] \in SO(4)$, $\nu = \nu(Y)$ and $T(Y) = \lambda_+(\nu)$. Also we can identify T^*Y with TY by the induced metric. From above we have the action $T^*Y \otimes \nu \to \nu$ inducing actions $\Lambda^*(T^*Y) \otimes \nu \to \nu$.

Let $\mathbb{L} = \Lambda^3(\Xi) \to \tilde{M}$ be the determinant (real) line bundle. Recall that the definition (9) implies that χ maps every oriented 3-plane in $T_x(M)$ to its complementary subspace, so χ gives a bundle map $\mathbb{L} \to \mathbb{V}$ over $\tilde{M}$, which is a section of $\mathbb{L}^* \otimes \mathbb{V} \to \tilde{M}$. Since Ξ is oriented $\mathbb{L}$ is trivial, so χ actually gives a section

$$\chi = \chi_\varphi \in \Omega^0(\tilde{M}, \mathbb{V}) \tag{28}$$

Clearly $\tilde{M}_\varphi \subset \tilde{M}$ is the codimension 4 submanifold which is the zeros of this section. Associative submanifolds $Y \subset M$ are characterized by the condition $\chi|_{\tilde{Y}} = 0$, where $\tilde{Y} \subset \tilde{M}$ is the canonical lifting of Y. Similarly φ defines a map $\varphi : \tilde{M} \to \mathbb{R}$.

3.1. **Pseudo-associative submanifolds.**

Here we generalize associative submanifolds to a more flexible class of submanifolds. To do this we first generalize the notion of imbedded submanifolds.

Definition 6. *A Grassmann-framed 3-manifold in (M, φ) is a triple (Y^3, f, F), where $f : Y \hookrightarrow M$ is an imbedding, $F : Y \to \tilde{M}$, such that the following commute*

$$\begin{tikzcd} & \tilde{M} \arrow[d] \\ Y \arrow[ur, "F"] \arrow[r, "f"] & M \end{tikzcd} \tag{29}$$

We call (Y, f, F) a pseudo-associative submanifold if in addition $Image(F) \subset \tilde{M}_\varphi$. So a pseudo-associative submanifold (Y, f, F) with $F = \tilde{f}$ is associative.

Remark 2. *The bundle $\tilde{M} \to M$ always admits a section, in fact the subbundle $\tilde{M}_\varphi \to M$ has a section. This is because by* [T] *every orientable* 7*-manifold admits a non-vanishing linearly independent* 2*-frame field $\Lambda = \{v_1, v_2\}^1$. By Grahm-Schmidt process with metric g_φ, we can assume that Λ is orthonormal. The cross product assigns Λ to an orthonormal* 3*-frame field $\{v_1, v_2, v_1 \times_\varphi v_2\}$ on M, then* 3*-plane generated by $\{v_1, v_2, v_1 \times_\varphi v_2\}$* $:=< v_1, v_2, v_1 \times_\varphi v_2 >$ gives a section of $\lambda_\varphi : M \to \tilde{M}_\varphi$. Let

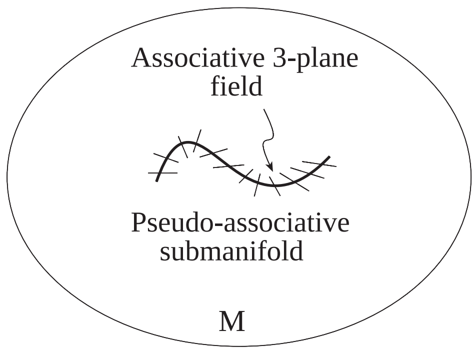

FIGURE 1.

$\mathcal{Z}(M)$ and $\mathcal{Z}_\varphi(M)$ denote the set of Grassmann-framed and the pseudo-associative submanifolds, respectively, and let $\mathcal{A}_\varphi(M)$ be the set of associative submanifolds. We have inclusions $\mathcal{A}_\varphi(M) \hookrightarrow \mathcal{Z}_\varphi(M) \hookrightarrow \mathcal{Z}(M)$, where the first map is given by $(Y, f) \mapsto (Y, f, \tilde{f})$. So there is an inclusion $Im(Y, M) \hookrightarrow \mathcal{Z}(M)$, where $Im(Y, M)$ is the space of imbeddings. This inclusion can be thought of the canonical sections of a bundle

$$\mathcal{Z}(Y) \xrightarrow{\pi} Im(Y, M) \tag{30}$$

with fibers $\pi^{-1}(f) = \Omega^0(Y, f^*\tilde{M})$. We also have the subbundle $\mathcal{Z}_\varphi(Y) \xrightarrow{\pi} Im(Y, M)$ with fibers $\pi^{-1}(f) = \Omega^0(Y, f^*\tilde{M}_\varphi)$. So $\mathcal{Z}(Y)$ is the set of triples (Y, f, F) (in short just set of F's), where $F : Y \to \tilde{M}$ is a lifting of the imbedding $f : Y \hookrightarrow M$. Also $\mathcal{Z}_\varphi(Y) \subset \mathcal{Z}(Y)$ is a smooth submanifold, since $\tilde{M}_\varphi \subset \tilde{M}$ is smooth. There is the canonical section $\Phi : Im(Y, M) \to \mathcal{Z}(Y)$ given by $\Phi(f) = \tilde{f}$. Therefore, $\Phi^{-1}\mathcal{Z}_\varphi(Y) := Im_\varphi(Y, M)$ is the set of associative imbeddings $Y \subset M$. Also, any 2-frame field Λ as above gives to a section $\Phi_\Lambda(f) = \lambda_\varphi \circ f$. To make these definitions parameter free we also have to divide $Im(Y, M)$ by the diffeomorphism group of Y.

[1] We thank T.Onder for pointing out [T]

There are also the vertical tangent bundles of $\mathcal{Z}(Y)$ and $\mathcal{Z}_{\varphi}(Y)$

$$\begin{array}{ccc} T^v\mathcal{Z}(Y) & \xrightarrow{\pi} & \mathcal{Z}(Y) \\ \cup & & \cup \\ T^v\mathcal{Z}_{\varphi}(Y) & \xrightarrow{\pi|} & \mathcal{Z}_{\varphi}(Y) \end{array}$$

with fibers $\pi^{-1}(F) = \Omega^0(Y, F^*(\Xi^* \otimes \mathbb{V}))$. By Lemma 3 the fibers of $T^v(\mathcal{Z}_{\varphi})$ can be identified with the kernel of the map induced by the Clifford multiplication

$$c : \Omega^0(Y, F^*(\Xi^* \otimes \mathbb{V})) \to \Omega^0(Y, F^*(\mathbb{V})) \tag{31}$$

One of the nice properties of a pseudo-associative submanifold (Y, f, F) is that there is a Clifford multiplication action (by pull back)

$$F^*(\Xi^*) \otimes F^*(\mathbb{V}) \to F^*(\mathbb{V}) \tag{32}$$

If F is close to $\tilde{f}$, by parallel translating the fibers over $F(x)$ and $\tilde{f}(x)$ along geodesics in $\tilde{M}$ we get canonical identifications:

$$F^*(\Xi) \cong TY \quad F^*(\mathbb{V}) \cong \nu_f \tag{33}$$

inducing Clifford multiplication between the tangent and the normal bundles. So if $\forall x \in Y$ the distance between $F(x)$ and $\tilde{f}(x)$ is less then the injectivity radius $j(\tilde{M})$, there is a Clifford multiplication between the tangent and normal bundles of Y.

3.2. **Dirac operator.**

The normal bundle $\nu = \nu(Y)$ of any orientable 3-manifold Y in a G_2 manifold (M, φ) has a $Spin(4)$ structure (e.g. [B2]). Hence we have $SU(2)$ bundles S and E over Y such that $\nu = S \otimes_{\mathbb{H}} E$ (18), with $SO(3)$ reductions $adS = \lambda_+(\nu)$, and $adE = \lambda_-(\nu)$ which is also the bundle of endomorphisms $End(E)$. If Y is associative, then the bundle $ad(S)$ becomes isomorphic to TY, i.e. S becomes the spinor bundle of Y, so $\nu(Y)$ becomes a twisted spinor bundle.

The Levi-Civita connection of the G_2 metric of (M, φ) induces connections on the associated bundles $\mathbb{V}$ and Ξ on $\tilde{M}$. In particular, it induces connections on the tangent and normal bundles of any submanifold $Y^3 \subset M$. We will call these connections the *background connections*. Let $\mathbb{A}_0$ be the induced connection on the normal bundle $\nu = S \otimes E$. From the Lie algebra decomposition $so(4) = so(3) \oplus so(3)$, we can write $\mathbb{A}_0 = B_0 \oplus A_0$, where B_0 and A_0 are connections on S and E, respectively.

Let $\mathcal{A}(E)$ and $\mathcal{A}(S)$ be the set of connections on the bundles E and S. Hence $A \in \mathcal{A}(E)$, $B \in \mathcal{A}(S)$ are in the form $A = A_0 + a$, $B = B_0 + b$, where $a \in \Omega^1(Y, ad\ E)$ and $b \in \Omega^1(Y, ad\ S)$. So $\Omega^1(Y, \lambda_{\pm}(\nu))$ parametrizes connections on S and E, and the connections on ν are in the form $\mathbb{A} = B \oplus A$. To emphasize the dependency on b and a we sometimes denote $\mathbb{A} = \mathbb{A}(b, a)$, and $\mathbb{A}_0 = \mathbb{A}(0, 0) = A_0$.

Now, let $Y^3 \subset M$ be any smooth manifold. We can ex press the covariant derivative $\nabla_{\mathbb{A}} : \Omega^0(Y,\nu) \to \Omega^1(Y,\nu)$ on ν by $\nabla_A = \sum e^i \otimes \nabla_{e_i}$, where $\{e_i\}$ and $\{e^i\}$ are orthonormal tangent and cotangent frame fields of Y, respectively. Furthermore, if Y is an associative submanifold, we can use the Clifford multiplication of (25) (i.e. the cross product) to form the twisted Dirac operator $\not{D}_{\mathbb{A}} : \Omega^0(Y,\nu) \to \Omega^0(Y,\nu)$

$$\not{D}_{\mathbb{A}} = \sum e^i \,.\nabla_{e_i} \tag{34}$$

The sections lying in the kernel of this operator are usually called harmonic spinors twisted by $(E,\mathbb{A})$. Elements of the kernel of $\not{D}_{A_0}$ are called the harmonic spinors twisted by E, or just the twisted harmonic spinors.

4. Deformations

In [M], McLean showed that the space of associative submanifolds of a G_2 manifold (M,φ), in a neighborhood of a fixed associative submanifold Y, can be identified with the harmonic spinors on Y twisted by E. Since the cokernel of the Dirac operator can vary, the dimension of its kernel is not determined (it has zero index since Y is odd dimensional). We will remedy this problem by deforming Y in a larger class of submanifolds. To motivate our aproach we will first sketch a proof of McLean's theorem (adapting the explanation in [B3]). Let $Y \subset M$ be an associative submanifold, Y will determine a lifting $\tilde{Y} \subset \tilde{M}_\varphi$. Let us recall that the G_2 structure φ gives a metric connection on M, hence it gives a connection A_0 and a covariant differentiation in the normal bundle $\nu(Y)=\nu$

$$\nabla_{A_0} : \Omega^0(Y,\nu) \to \Omega^1(Y,\nu) = \Omega^0(Y,\ T^*Y \otimes \nu)$$

Recall that we identified $T^*_y(Y)\otimes \nu_y(Y)$ by the tangent space of the Grassmannian of 3-planes $TG(3,7)$ in $T_y(M)$. So the covariant derivative lifts normal vector fields v of $Y \subset M$ to vertical vector fields $\tilde{v}$ in $T(\tilde{M})|_{\tilde{Y}}$. We want the normal vector fields v of Y to move Y in the class of associative submanifolds of M, i.e. we want the liftings $\tilde{Y}_v$ of the nearby copies Y_v of Y (pushed off by the vector field v) to lie in $\tilde{M}_\varphi \subset \tilde{M}$ upstairs, i.e. we want the component of $\tilde{v}$ in the direction of the normal bundle $\tilde{M}_\varphi \subset \tilde{M}$ to vanish. By Lemma 3, this means $\nabla_{A_0}(v)$ should be in the kernel of the Clifford multiplication $c = c_\varphi : \Omega^0(T^*(Y)\otimes\nu) \to \Omega^0(\nu)$, i.e. $\not{D}_{A_0}(v) = c(\nabla_{A_0}(v)) = 0$, where $\not{D}_{A_0}$ is the Dirac operator induced by the background connection A_0, i.e. the composition

$$\Omega^0(Y,\nu) \xrightarrow{\nabla_{A_0}} \Omega^0(Y,T^*Y\otimes\nu) \xrightarrow{c} \Omega^0(Y,\nu) \tag{35}$$

The condition $\not{D}_{A_0}(v)=0$ implies φ must be integrable at Y, i.e. the $so(7)$-metric connection ∇_{A_0} on Y coincides with G_2-connection (c.f. [B2]).

Now we give a general version of the McLean's theorem, without integrability assumption on φ: Recall from (Section 3.1) that $\Phi^{-1}\mathcal{Z}_\varphi(Y)$ is the set of associative

submanifolds $Y \subset M$, where $\Phi : Im(Y,M) \to \mathcal{Z}(Y)$ is the canonical section (Gauss map) given by $\Phi(f) = \tilde{f}$. Therefore, if $f : Y \hookrightarrow M$ is the above inclusion, then $\Phi(f) \in \mathcal{Z}_{\varphi}$. So this moduli space is smooth if Φ was transversal to $\mathcal{Z}_{\varphi}(Y)$.

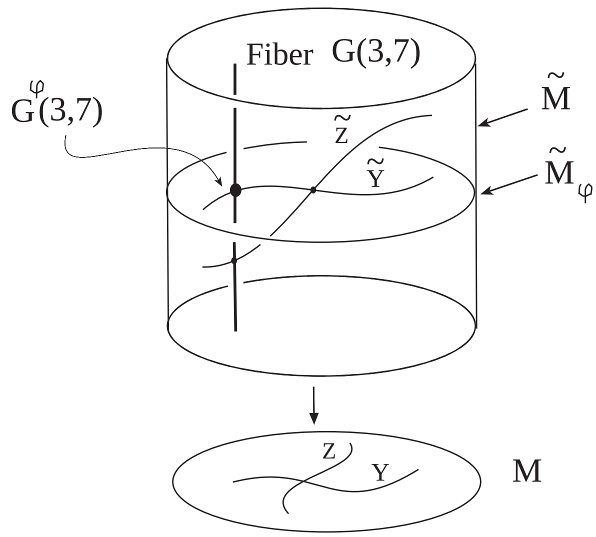

FIGURE 2.

Theorem 4. *Let (M^7, φ) be a manifold with a G_2 structure, and $Y^3 \subset M$ be an associative submanifold. Then the tangent space of associative submanifolds of M at Y can be identified with the kernel of a Dirac operator $\not{D}_A : \Omega^0(Y,\nu) \to \Omega^0(Y,\nu)$, where $A = A_0 + a$, and A_0 is the connection on ν induced by the metric g_{φ}, and $a \in \Omega^1(Y, ad(\nu))$. In the case φ is integrable $a = 0$. In particular, the space of associative submanifolds of M is smooth at Y if the cokernel of $\not{D}_A$ is zero.*

Proof. Let $f : Y \hookrightarrow M$ denote the imbedding. We consider unparameterized deformations of Y in $Im(Y,M)$ along its normal directions. Fix a trivialization $TY \cong im(\mathbb{H})$, by (17) we have an identification $\tilde{f}^*(T^v\tilde{M}) \cong TY^* \otimes \nu + TY + \nu$. We first claim $\Pi \circ d\Phi(v) = \nabla_A(v)$, where $d\Phi$ is the induced map on the tangent space and Π is the vertical projection.

$$\begin{array}{ccc} \Omega^0(Y,\nu) = T_f Im(Y,M) & \xrightarrow{d\Phi} & T_{\tilde{f}}\mathcal{Z}(Y) = \Omega^0(Y, \tilde{f}^*(T^v\tilde{M})) \xrightarrow{\Pi} \Omega^0(Y, T^*Y \otimes \nu) \\ \downarrow exp & & \downarrow exp \\ Im(Y,M) & \xrightarrow{\Phi} & \mathcal{Z}(Y) \end{array}$$

The two vertical maps $v \to f_v$, and $w \to (\tilde{f})_w$ are exponential projections of tangent vectors, i.e. $f_v(y) = exp_{f(y)}(v)$ and $(\tilde{f})_w(y) = exp_{\tilde{f}(y)}(w)$. It suffices to check this claim pointwise. Here for convenience view f as an inclusion $Y \subset M$.

Let $y = (y_1, y_2, y_3)$ be the normal coordinates of Y centered around y_0, and $\{e_j\}_{j=1}^{7}$ be an orthonormal frame field of M defined on Y, with $e_j(y_0) = \partial/\partial y_j$ for $j = 1, 2, 3$. To this data we can associate *Fermi coordinates* (y, t) around $f(y_0) \in M$ (they are a version of normal coordinates along a submanifold, see for example [G]):

$$(y, t) \longleftrightarrow f_{\sum t_\alpha e_\alpha}(y) \tag{36}$$

where $t = (t_4, .., t_7)$. Then we can write $\tilde{f}(y_0) = e_1 \wedge e_2 \wedge e_3$. Hence by definition we can express $d\Phi(v) = \widetilde{(f_v)} = (f_v)_*(e_1) \wedge (f_v)_*(e_2) \wedge (f_v)_*(e_3) := e_1(v) \wedge e_2(v) \wedge e_3(v)$.

$$d\Phi(v)(y_0) = \mathcal{L}_v(e_1 \wedge e_2 \wedge e_3) = \sum_{i=1}^{3} (*e_j) \wedge \mathcal{L}_v(e_j)|_Y \tag{37}$$

where $\mathcal{L}_v$ denotes Lie derivative along v, and $*$ is the star of Y. The metric connection is torsion free hence $\mathcal{L}_v(e_j) = \bar{\nabla}_{e_j}(v) - \bar{\nabla}_v(e_j)$, where $\bar{\nabla}$ is the metric connection of M. In case (M, φ) is a G_2 manifold (i.e. when φ integrable), by (2) and (7), up to quadratic term φ is φ_0, therefore we can write:

$$0 = \bar{\nabla}_v(\varphi)|_Y = \bar{\nabla}_v(e^1 \wedge e^2 \wedge e^3)|_Y = \sum_{j=1}^{3} (*e^j) \wedge \bar{\nabla}_v(e^j)|_Y \ , \quad \text{which implies}$$

$$\Pi \circ d\Phi(v)(y_0) = \sum_j (*e_j) \wedge \nabla_{e_j}(v) \tag{38}$$

where where $\{e^j\}$ is the dual coframe, and $\nabla_{e_j}(v)$ is the normal component of $\bar{\nabla}_{e_j}(v)$, i.e. it is the induced connection on $\nu(Y)$. The expression (38) can be viewed as an infinitesimal deformation of the 3-plane $\tilde{f}(y_0)$. By the identification $*e_j \leftrightarrow e^j$ we can view it as an element of the tangent space $T^*Y \otimes \nu$ of the Grassmannian of 3-planes in $T_{y_0}(M)$

$$\Pi \circ d\Phi(v)(y_0) = \sum e^j \otimes \nabla_{e_j}(v)(y_0) = \nabla_{A_0}(v)(y_0) \tag{39}$$

When φ is not integrable, there is an extra term which we can write

$$\sum (*e_j) \wedge \bar{\nabla}_v(e_j) = \sum (*e_j) \wedge \nabla_v(e_j)$$

where $\nabla_v(e_j)$ is the normal component of $\bar{\nabla}_v(e_j)$. Notice $< \bar{\nabla}_v(e_k), e_k >= 0$, which is implied by $v < e_k, e_k >= 0$. So in this case (39) becomes

$$\Pi \circ d\Phi(v)(y_0) = \nabla_{A_0}(v) + a(v) = \nabla_A(v) \tag{40}$$

where $a(v) = \sum e^j \otimes \nabla_v(e_j) \in \Omega^1(Y, ad(\nu))$ and $A = A_0 + a$. It easy to check that the expression $a(v)$ is independent of the choice the orthonormal frame $\{e_j\}$.

By (31) the vertical tangent space of $\mathcal{Z}_{\varphi}(Y)$ is given by the kernel of the Clifford multiplication $c_{\varphi}:\Omega^0(T^*Y\otimes\nu)\to\Omega^0(\nu)$. So, locally the moduli space of associative submanifolds of (M,φ) is given by the kernel of $\not{D}_A$, i.e. the condition that $d\Phi(v)$ lies in $T^v_{\tilde f}\mathcal{Z}_{\varphi}(Y)$ is given by $\not{D}_A(v)=0$. The moduli space is smooth if Φ is transversal to $\mathcal{Z}_{\varphi}(Y)$, i.e. if the cokernel of $\not{D}_A$ is zero. Since $T^v_{\tilde f}\mathcal{Z}(Y)=\Omega^0(T^*Y\otimes\nu)$ and

$$T\mathcal{Z}(Y)/T\mathcal{Z}_{\varphi}(Y)=T^v\mathcal{Z}(Y)/T^v\mathcal{Z}_{\varphi}(Y)$$

to check transversality we look at the induced maps, and use $\Pi\circ d\Phi_f(v)=\nabla_A(v)$

$$\Omega^0(\nu)=T_f Im(Y,M)\xrightarrow{d\Phi}T_{\tilde f}\mathcal{Z}(Y)\xrightarrow{\Pi}T^v_{\tilde f}\mathcal{Z}(Y)\supset T^v_{\tilde f}\mathcal{Z}_{\varphi}(Y)\qquad\square$$

Remark 3. *This theorem can also be proved by generalizing McLean's proof: The condition that an associative $Y\subset M$ remains associative, when moved via the exponential map along a normal vector field $v\in\Omega^0(Y,\nu)$, is $\mathcal{L}_v(\chi)|_Y=0$. We can choose local coordinates $(x_1,...,x_7)$ on M, such that (x_1,x_2,x_3) gives the coordinates of Y. By (13) and (14) $\chi=\sum a^{\alpha}_J\,dx^J\otimes\partial/\partial x_{\alpha}$, with $\alpha\notin J$ and $a^{\alpha}_{123}|_Y=0$*

$$\mathcal{L}_v(\chi)|_Y=\sum v(a^{\alpha}_{123})\frac{\partial}{\partial x_{\alpha}}+\sum a^{\alpha}_J\,\mathcal{L}_v(dx^J)|_Y\otimes\frac{\partial}{\partial x_{\alpha}}=0 \tag{41}$$

McLean treated integrable φ case, i.e when (M,φ) is a G_2 manifold. In this case the first term vanishes, and the second becomes $\not{D}_{A_0}(v)\otimes dx^{123}$. But notice that the first term $a(v)$ is linear in v and takes values in $\Omega^0(Y,\nu)$, hence $a\in\Omega^0(Y,ad(\nu))$. So in the non-integrable case we get a twisted Dirac equation $\not{D}_A(v)=0$, where $A=A_0+a$.

If in the proof of Theorem 4 we replace the G_2 structure φ with another G_2 structure ψ inducing the same metric, the identification of the bundle $TY^{\perp}=\nu$ doesn't change but the Clifford action c_{φ} changes to another one c_{ψ}, corresponding to another 4-dimensional Clifford represention of T^*Y. These two representations are conjugate by a gauge automorphism γ of ν.

$$\begin{array}{ccc}\Omega^0(T^*Y\otimes\nu) & \xrightarrow{c_{\psi}} & \Omega^0(\nu)\\ 1\otimes\gamma\ \downarrow & & \gamma\ \downarrow\\ \Omega^0(T^*Y\otimes\nu) & \xrightarrow{c_{\varphi}} & \Omega^0(\nu)\end{array}$$

Therefore, if we call the Dirac operator induced by ψ by $\not{D}_{A_1}$, we can write

$$\gamma(\not{D}_{A_1}(w))=\sum dy^j.\gamma(\nabla_j(w))=\sum dy^j.(\nabla_j\gamma(w))-dy^j.(\nabla_j\gamma)(w)$$

where the dot "." denotes the Clifford product c_{φ}. So, $D_{A_1}(w)=0$ gives a twisted version of the Dirac equation $D_{A_0}(v)=0$ where $v=\gamma(w)$, this is because $\gamma(\not{D}_{A_1}(w))=\not{D}_{A_0+a}(\gamma(w))$, where $a=-\sum dy^j.(\nabla_j\gamma)\gamma^{-1}$. In Theorem 6 we will use the twisting of the Dirac operator, under deformations of φ, to obtain its surjectivity.

5. Transversality

We can make the cokernel of Dirac operator $\not{D}_{A_0}$ zero either by deforming the Gauss map $\Phi : Im(Y,M) \to \mathcal{Z}(Y)$, or by deforming the G_2 structure φ. Changing φ can be realized by deforming φ by a gauge transformation: Recall that the G_2 structures φ on M are the sections $\Omega^3_+(M)$ of the bundle (5). Also $GL(7,\mathbb{R})$ conjugates $G_2 = G_2^{\varphi_0}$ to any other G_2 subgroup G_2^{φ} of $GL(7,\mathbb{R})$ where

$$G_2^{\varphi} = \{A \in GL(7,\mathbb{R}) \mid A^*\varphi = \varphi\} \overset{\varphi}{\hookrightarrow} SO(7)$$

If we are interested in the G_2 structures inducing the same metric, we replace $GL(7,\mathbb{R})$ with $SO(7)$. $SO(7)$ acts on $G(3,7)$ permuting submanifolds $G^{\varphi}(3,7)$, where $\varphi \in \Omega^3_+(M)$. More generally the gauge group $\mathcal{G}(P)$ of $P = P_{SO(7)} \to M$ acts on $\tilde{M}$ permuting $\tilde{M}_{\varphi}$'s. Recall that $\mathcal{G}(P) = \{P \overset{s}{\to} P \mid s(pg) = s(p)g\ \}$, which can be identified with sections $\Omega^0(M; Ad(P))$ of the bundle $Ad(P) \to M$ (c.f. [AMR]), where

$$Ad(P) = P \times_{Ad} SO(7) = \{[p,h] \mid (p,h) \sim (pg, g^{-1}hg)\}$$

$$\text{One can also identify: } \ \mathcal{G}(P) = \{s : P \to SO(7) \mid s(pg) = g^{-1}s(p)g\ \}$$

The tangent space of $\mathcal{G}(P)$ at the identity I are the sections $\mathfrak{g}(P) = \Omega^0(M, ad(P))$ of the associated bundle of Lie algebras $ad(P) = P \times_{Ad} so(7) \to M$. Similarly

$$\mathfrak{g}(P) = \{h : P \to so(7) \mid h(pg) = h(p)g - gh(p)\ \}$$

We can identify $T_s(\mathcal{G}_P(M)) \overset{\cong}{\mapsto} \mathfrak{g}(P)$, by $s \mapsto s^{-1}ds$. There is also an action $\mathcal{G}(P) \times \tilde{M} \to \tilde{M}$ given by $(s, [p,L]) \to s[p,L] := [ps(p), s(p)L]$, which we will simply denote it by $(s,L) \mapsto s.L$. There is the pull-back action $\mathcal{G}(P) \times \Omega^3_+(M) \to \Omega^3_+(M)$ given by $(s,\varphi) \to s^*(\varphi)$. In particular, $s\tilde{M}_{\varphi} = \tilde{M}_{s^*\varphi}$. Put another way, if $\chi = \chi_{\varphi}$ is the 3-form of Definition 3 and $L \in \tilde{M}$, then $\chi|_L = 0 \iff s^*\chi|_{s^{-1}L} = 0$. Hence the 3-plane sL is φ-associative $\iff$ L is $s^*\varphi$ -associative (similar to the process in the Kleiman transversality, c.f. [AK])

From the above action, we see that the space of G_2 structures $\tilde{\Omega}^3_+(M)$ which induce the same metric on M has the following identification:

Lemma 5. *Let $\mathcal{G}(P_{G_2})$ be the stabilizer of the action of $\mathcal{G}(P)$ on $\tilde{\Omega}^3_+(M)$ (i.e. the gauge transformations fixing φ) then:*

$$\tilde{\Omega}^3_+(M) = \mathcal{G}(P)/\mathcal{G}(P_{G_2}) = \Omega^0(M, P \times_{SO(7)} \mathbb{RP}^7)$$

Proof. Clearly $\mathcal{G}(P)$ acts transitively on $\tilde{\Omega}^3_+(M)$ with stabilizer $\mathcal{G}(P_{G_2})$. To see the second equality, we identify the fibers of the coset space with the fibers of $\tilde{\Lambda}^3_+(M) \to M$ by the map:

$$\mathbb{RP}^7 = SO(7)/G_2^{\varphi} \to \tilde{\Lambda}^3_+(\mathbb{R}^7)$$

$G_2^{\varphi}s \mapsto s^*\varphi$. The adjoint action of $SO(7)$ on $SO(7)$ moves cosets

$$G_2^{\varphi}s \mapsto (g^{-1}G_2^{\varphi}g)g^{-1}sg = G^{g^*\varphi}g^{-1}sg$$

Hence by the obove identification, on $\mathbb{RP}^7$ it induces $\varphi \mapsto g^*\varphi$. $\square$

Now we can deform the canonical section $\Phi : Im(Y,M) \to \mathcal{Z}(Y)$ by the map

$$\tilde{\Phi} : \mathcal{G}(P) \times Im(Y,M) \to \mathcal{Z}(Y) \tag{42}$$

$\tilde{\Phi}(s,f) = \Phi_s(f) = s\Phi(f) = s(\tilde{f})$, that is $\tilde{\Phi}(s,f)(y) = s(f(y))\tilde{f}(y)$. Notice $\mathcal{G}(P)$ acts on the sections of the bundle $\mathcal{Z}(Y) \to Im(Y,M)$.

Theorem 6. *$\tilde{\Phi}$ is transversal to $\mathcal{Z}_{\varphi}(Y)$. Also Φ_s is transversal to $\mathcal{Z}_{\varphi}(Y)$ for a generic choice of s, equivalently Φ is transversal to $\mathcal{Z}_{s^*\varphi}(Y)$ for a generic s.*

Proof. : Let $\tilde{\Phi}(s,f) \in \mathcal{Z}_{\varphi}(Y)$. We can check transversality of $\tilde{\Phi}$ at (s,f) by computing its derivative. By the Leibnitz rule and Theorem 4 we can compute

$$s^{-1} \circ \Pi \circ d\tilde{\Phi}(h,v) : \mathfrak{g}_P(M) \oplus \Omega^0(\nu) \to T^v_{s(\tilde{f})}\mathcal{Z}(Y) \to T^v_{\tilde{f}}\mathcal{Z}(Y) = \Omega^0(T^*Y \otimes \nu)$$

where $d\tilde{\Phi}\ (h,v) = s(f)\ [\nabla_{A_0}(v) + s^{-1}ds(v)\tilde{f}\ ]$, and $v = f_v$ is the perturbation of the inclusion f. Observe that $ad(P) = End(TM)$, and the map $y \mapsto s^{-1}ds(v)\tilde{f}(y)$ is a vertical deformation of the 3-plane $y \mapsto \tilde{f}(y) = T_yY$, hence it is a section of the pull back of the vertical tangent bundle of $\tilde{M} \to M$ over Y, i.e. an element $a(v) \in T^v_{\tilde{f}}\mathcal{Z}(Y) = \Omega^0(T^*Y \otimes \nu)$. More specifically, if we decompose $s^{-1}ds(v)$ as an element of $so(7)$ on $T_{f(y)}(M) = T_yY \oplus \nu_y(Y)$ in block matrices we can write:

$$s^{-1}ds(v)\mid_{f(y)} = \begin{pmatrix} * & -\alpha(v)^t \\ \alpha(v) & * \end{pmatrix} \tag{43}$$

Because $\alpha(v)$ is linear in v, we can view $\alpha \in \Omega^1(Y, ad\ \nu)$, therefore we can express $s^{-1}\Pi \circ d\Phi_s(v) = \nabla_{A_0}(v) + \alpha(v) = \nabla_{\mathbb{A}}(v)$ with $\mathbb{A} = A_0 + \alpha$. So the transversality is measured by the cokernel of the twisted Dirac operator $c_{\varphi}(\nabla_{\mathbb{A}}) = \not{D}_{\mathbb{A}}$, where c_{φ} is the Clifford multiplication. Now by choosing $\alpha(v)$ we show that we can make $\not{D}_{\mathbb{A}}$ onto. This is because, if $\not{D}_{A_0}$ is not already onto, we choose $0 \neq w \in im(\not{D}_{A_0})^{\perp}$. By self adjointness of the Dirac operator $0 =< \not{D}_{A_0}(v), w >=< v, \not{D}_{A_0}(w) >$, for all v. So $\not{D}_{A_0}(w) = 0$, by analytic continuation $w \neq 0$ on an open set. Then $w \in im(\not{D}_{\mathbb{A}})^{\perp}$ implies $< c_{\varphi}(\alpha(v)), w >= 0$ and hence $w = 0$, which is a contradiction. The last implication follows from by choosing s in (43) we can get the full Lie algebra $so(7)$, and hence $v \mapsto a(v)$ is onto, and the Clifford multiplication c is onto (Lemma 3).

So we obtain a smooth manifold $\tilde{\Phi}^{-1}\mathcal{Z}_{\varphi}(Y)$, and by choosing a regular value s of the projection $\tilde{\Phi}^{-1}\mathcal{Z}_{\varphi}(Y) \to \mathcal{G}_P(M)$ we get $\tilde{\Phi}_s^{-1}\mathcal{Z}_{\varphi}(Y)$ smooth (note that the derivative of the projection is Fredholm). Clearly the condition that Φ_s transversal to $\mathcal{Z}_{\varphi}(Y)$ is equivalent to Φ being transversal to $\mathcal{Z}_{s^*\varphi}(Y)$. $\square$

Theorem 6 says that the space of $s^*\varphi$ associative deformations of an φ associative submanifold $Y \subset M$, where $s \in \mathcal{G}_P(M)$, is a smooth (infinite dimensional) manifold. Infinitesimally these deformations correspond to the kernel of the twisted Dirac operator, twisted by the connections in the normal bundle $\nu(Y)$. Define

$$\sigma : \mathcal{G}(P) \to \Omega^0(\tilde{M}, \Xi^* \otimes \lambda(\mathbb{V})) \tag{44}$$

by $\sigma(s)(L)(v) = \alpha_s(v, L) \in \Xi^* \otimes \mathbb{V}$, where $\alpha_s(v, L)$ is obtained by decomposing $s^{-1}ds(v) \in \Omega^0(M, ad(P))$ on $TM = L \oplus L^{\perp}$ as an element of $so(7)$

$$s^{-1}ds(v) \mid_L = \begin{pmatrix} * & -\alpha(v,L)^t \\ \alpha(v,L) & * \end{pmatrix} \tag{45}$$

We can think of $\Omega^0(\tilde{M}, \Xi^* \otimes \lambda(\mathbb{V}))$ as an universal space parameterizing connections on $\nu \to Y$. The Gauss map $\tilde{f}$ of any imbedding $f : Y \hookrightarrow M$ pulls back $\Xi^* \otimes \lambda(\mathbb{V})$ to the parameter space $\Omega^1(Y, \lambda(\nu))$ of the connections on $\nu(Y)$.

$$\Omega^0(\tilde{M}, \Xi^* \otimes \lambda(\mathbb{V})) \xrightarrow{\tilde{f}^*} \Omega^1(Y, \lambda(\nu)) \tag{46}$$

Clearly the set $\Omega^1(\tilde{M}, \lambda(\mathbb{V}))$ can also be used as the universal parameter space. As in Section 3.2, given any imbedding $f : Y \hookrightarrow M$, we can deform the background connection $A_0 \to A = \mathbb{A}(b, a)$ in the normal bundle $\nu(Y)$, with $b \in \Omega^1(Y, \lambda_+(\nu))$ and $a \in \Omega^1(Y, \lambda_-(\nu))$, and get a perturbed version of (35)

$$\Omega^0(\nu) \times \Omega^1(\lambda_\pm(\nu)) \xrightarrow{\not{D}_A} \Omega^0(\nu) \tag{47}$$

with the twisted Dirac equation $\not{D}_{\mathbb{A}}(v) = c(\nabla_{\mathbb{A}}(v)) = \not{D}_{A_0}(v) + \alpha v = 0$, where $\alpha = (b, a)$. Here we prefer perturbing by a (perturbing b has the effect of perturbing the metric on Y). A generic nonzero a makes the map $v \mapsto \not{D}_{A_0+a}(v)$ surjective. We can choose this perturbation term a universally.

6. Complex Associative Submanifolds

Let (M, φ) be a manifold with a G_2 structure. Here we will study an interesting class of associative submanifolds whose normal bundles come with an almost complex structure. The subgroups $U(2) \subset SO(4) \subset G_2 = G_2^{\varphi}$, more specifically

$$(S^1 \times SU(2))/\mathbb{Z}_2 \subset (SU(2) \times SU(2))/\mathbb{Z}_2 \subset G_2$$

give a $U(2)$-principal bundle $\mathcal{P}_{G_2}(M) \to \bar{M}_\varphi = \mathcal{P}_{G_2}(M)/U(2)$. Also $\bar{M}_\varphi$ is the total space of an S^2 bundle $\bar{M}_\varphi \to \tilde{M}_\varphi = \mathcal{P}_{G_2}(M)/SO(4)$, which is just the sphere bundle

$$\bar{M}_\varphi = S(\Xi) \to \tilde{M}_\varphi \tag{48}$$

of the $\mathbb{R}^3$-bundle: $\lambda_+(\mathbb{V}) = \Xi \to \tilde{M}_\varphi$. We can identify the sections of (48) with almost complex structures on $\mathbb{V}$. Notice $\bar{M}_\varphi \to M$ is a bundle with fibers $G_2/U(2)$, which we can view as the complex version of the associative Grassmanns $G_{\mathbb{C}}^{\varphi}(3, 7)$.

In fact if $V_2(M)$ and $G_2(M)$ are the bundle of orthonormal 2-frames and oriented 2-planes in M respectively, the fibration $V_2(M) \to G_2(M)$ can be identified by:

$$\mathcal{P}_{G_2}(M)/SU(2) \to \mathcal{P}_{G_2}(M)/U(2)$$

with its projection $\{u, v\} \mapsto (< u, v, u \times v >, u \times v)$, and also the projection map $\mathcal{P}_{G_2}(M) \to \mathcal{P}_{G_2}(M)/SU(2)$ on the fibers is given by the map $G_2 \to V_2(\mathbb{R}^7)$ defined by $\{v_1, v_2, v_3\} \mapsto \{v_1, v_2\}$ (recall the definition of G_2 in (1)). Put another way, G_2 acts transitively on $V_2(\mathbb{R}^7)$ with stabilizer $SU(2)$. By summing up above:

Proposition 7. $\bar{M}_\varphi = S(\Xi) = G_2(M)$

More generally, for the Riemannian manifold (M^7, g_φ) we can take the sphere bundle $\bar{M} \to \tilde{M}$ of $\lambda_+(\mathbb{V}) \to \tilde{M}$, and get codimension 4 inclusion of the smooth manifolds $\bar{M}_\varphi \subset \bar{M}$ (of dimensions 17 and 21). The sections of the bundle $\bar{M} \to \tilde{M}$ gives the parametrization of the almost complex structures on $\mathbb{V}$, and $\bar{M} \to M$ is a bundle with fibers $G_{\mathbb{C}}(3,7) := SO(7)/U(2) \times SO(3)$. For all G_2 structures φ inducing the same metric on M, we have the inclusions $G_2^\varphi \hookrightarrow SO(7)$ inducing imbeddings $G_2(M) = \bar{M}_\varphi \hookrightarrow \bar{M}$, which is fiberwise $< u, v > \mapsto < u, v, u \times v >$

$$G(2,7) = G_2^\varphi/U(2) \hookrightarrow SO(7)/U(2) \times SO(3)$$

By [T] the bundle $V_2(M) \to M$ has always a section $\Lambda = \{u, v\}$, which induces sections of the bundles $\bar{M}_\varphi \to \tilde{M}_\varphi$ and $\tilde{M}_\varphi \to M$ (for simplicity we will abuse notation and denote all these sections by Λ also). So Λ gives an almost complex structure on $\mathbb{V} \to \tilde{M}_\varphi$. By Λ, we can pull back Ξ and $\mathbb{V}$ to bundles $\mathbf{E}$ and $\mathbf{V}$ on M, respectively, and $\mathbf{V}$ has an almost complex structure (by the discussion following Lemma 2 we can describe this complex structure with the cross product with $u \times v$).

Definition 7. *From now on, we will denote a* 7*-manifold with a* G_2 *structure and a nonvanishing* 2*-frame field* Λ *with* (M, φ, Λ)*.*

Given (M, φ, Λ), then the induced $U(2)$ structure on $\mathbb{V} \to \tilde{M}_\varphi$ canonically lifts to a $Spin^c(4)$ structure by the diagram:

$$\begin{array}{ccc} & & Spin^c(4) \\ & \nearrow & \downarrow \\ U(2) & \rightarrow & SO(4) \times S^1 \end{array} \tag{49}$$

where $U(2) = (S^1 \times S^3)/\mathbb{Z}_2$, $SO(4) = (S^3 \times S^3)/\mathbb{Z}_2$, $Spin^c(4) = (S^3 \times S^3 \times S^1)/\mathbb{Z}_2$, where the horizontal map $[\lambda, A] \mapsto ([\lambda, A], \lambda^2)$ lifts to the map $[\lambda, A] \mapsto (\lambda, A, \lambda)$. This means there is a $\mathbb{C}^2$-bundle $\mathbb{W} \to \tilde{M}_\varphi$ with $\mathbb{V}_{\mathbb{C}} = \mathbb{W} \oplus \bar{\mathbb{W}}$, and transition function λ^2 gives the determinant line bundle $K = \Lambda^2 \bar{\mathbb{W}} \to \tilde{M}_\varphi$. Also we can write $\mathbb{V}_{\mathbb{C}} = \mathbb{W}^+ \otimes \mathbb{W}^-$ with $\mathbb{W}^+ = K^{-1} + \mathbb{C}$ and $\mathbb{W}^- = \bar{\mathbb{W}}$. Recall $\Xi^* = \lambda_+(\mathbb{V}) = \Lambda^2_+(\mathbb{V}) = K + \mathbb{R}$. We have a Clifford action $\mathbb{V} \otimes \mathbb{W}^+ \to \mathbb{W}^-$ which extends to Cllifford action

$$\Xi^* \otimes \mathbb{W}^+ \to \mathbb{W}^+$$

The identifications $\mathbb{W}^{+} \otimes \bar{\mathbb{W}}^{-} = \mathbb{C} \oplus \mathbb{C} \oplus K \oplus \bar{K} = (K \oplus \mathbb{R})_{\mathbb{C}} \oplus \mathbb{C} = \Xi^{*}_{\mathbb{C}} \oplus \mathbb{C}$ gives the usual quadratic bundle map of the Seiberg-Witten theory (c.f. [A]):

$$\sigma : \mathbb{W}^{+} \otimes \mathbb{W}^{+} \to \Xi^{*}_{\mathbb{C}} \tag{50}$$

$$\sigma(x,x) = (\frac{|z|^2 - |w|^2}{2}, \bar{z}w) \ \ , \ \ \text{where} \ \ x = (z,w)$$

Definition 8. *A submanifold $f : Y^3 \hookrightarrow M$ of (M^7, φ, Λ) is called Λ-associative if $\tilde{f} = \Lambda \circ f$ where $\tilde{f}$ is the Gauss map, and it is called almost Λ-associative if it comes from transverse section of the bundle $\mathbf{V} \to M$ (recall $\mathbf{V}$ is obtained from Λ).*

An Λ-associative, more generally almost Λ-associative submanifold Y of (M, φ, Λ) induce canonical isomorphisms $TY \cong \tilde{f}^{*}(\Xi)$ and $\nu(Y) \cong \tilde{f}^{*}(\mathbb{V})$ (by transversality).

$$\begin{array}{ccc} & & \tilde{M}_{\varphi} \\ & \tilde{f} \nearrow & \downarrow\uparrow \Lambda \\ Y & \stackrel{f}{\hookrightarrow} & M \end{array}$$

So normal bundle of any almost Λ-associative submanifold $Y^3 \subset M^7$ has a $U(2)$ structure, therefore it has a $Spin^c(4)$ structure, with induced $\mathbb{C}^2$-bundle $W \to Y$ and its determinant line bundle $K \to Y$, and a Clifford action of $T^{*}Y \otimes W \to W$ (induced from the cross product). An example of a Λ associative submanifold is the zero section of the spinor bundle $\mathcal{S} \to Y^3$ (the G_2 manifold constructed in [BSa]).

In general, the background $SO(4)$ connection A_0 on the normal bundle $\nu(Y)$ of a Λ-associative submanifold Y may not reduce to a $U(2)$ connection if the 1-form whose dual gives the splitting $TY \cong K \oplus \mathbb{R}$ is not parallel. Nevertheless from the $Spin^c(4)$ structure on $\nu(Y)$ we do get a connection on the complex bundle $W \to Y$ provided we pick a connection on the line bundle $K \to Y$ (from (49)). In the next section we will study the local deformation space of Λ-associative manifolds, by deforming them in the complex bundle W, with the help of the connections on K.

Remark 4. *Associative submanifolds with $Spin^c(3)$ structure $(Y, c) \hookrightarrow (M, \varphi)$ come equipped with an $U(2)$ structure (hence $Spin^c(4)$ structure) on their normal bundles (Lemma 2). We can free the deformations space these manifolds from the extra parameter c, by picking up a generic Λ, and studying the deformations of more relax almost Λ-associative submanifolds $Y \subset (M, \varphi, \Lambda)$. In this case the $Spin^c(3)$ structure on TY comes from the pull-back. Also, by further deforming the 2-plane field Λ on M deforms the the the $Spin^c$ structure on Y.*

Remark 5. *We could have considered complex structures on $\mathbb{V}$ corresponding to the right reduction, i.e. the subgroup $(SU(2) \times S^1)/\mathbb{Z}_2 \subset (SU(2) \times SU(2))/\mathbb{Z}_2 \subset G_2$. In this case, they correspond to the sections of the S^2 bundle $\lambda_{-}(\mathbb{V}) \to \tilde{M}_{\varphi}$. Here we opted to the left reductions since they concretely relate to Ξ by $\lambda_{+}(\mathbb{V}) = \Xi$.*

7. Deforming Λ-associative submanifolds

Let (M,φ,Λ) be a manifold with G_2 structure and a non-vanishing 2-plane field, $\mathcal{M}(M,\varphi,\Lambda)$ be the space of Λ-associative submanifolds. Here we will study the local "complex" deformations of $\mathcal{M}(M,\varphi,\Lambda)$ near a particular $f:Y\hookrightarrow M$. These are the deformations of Y inside its complex normal bundle W, with the help of the connections $\mathcal{A}(K)$ on the line bundle $K=$ det W. These deformations are identified with the kernel of a twisted Dirac operator twisted by the connections in $\mathcal{A}(K)$. Introducing new variables $\mathcal{A}(K)$ makes the deformation space smooth. Up to this point this section can be viewed as a version of Theorem 4 for the Λ-associative submanifolds. But now the connection parameter can be constraint with the natural map (50) to obtain Seiberg-Witten like equations, which gives a compactness result for this more restricted local deformation space of Y. Reader should note that these equations are $Spin^c(4)$ Seiberg-Witten equations on Y^3 (which are usually associated to 4-manifolds), as opposed to the usual $Spin^c(3)$ Seiberg-Witten equations. The Clifford action $T^*Y\otimes\mathbb{W}\to W$ is induced via the identification $\Lambda^2_+W=T^*Y$, it is also induced by the cross product operation on M.

Let $Y\in\mathcal{M}(M,\varphi,\Lambda)$. Let $W\to Y$ be the complex bundle associated to $\nu(Y)$, and $K\to Y$ be its determinant line bundle. Let B_0 be the background connection on $\nu(Y)$ (induced by φ), then as discussed in last section B_0 along with $A\in\mathcal{A}(K)$ defines a connection on $W\to Y$, denote by $\mathbb{A}=B_0\oplus A$. We can write $A=A_0+a$ with $a\in\Omega^1(Y)=T_{A_0}\mathcal{A}(K)$ (tangent space of connections) and $\mathbb{A}=\mathbb{A}(a)$. Then we get a complex version of the map (47) $(v,a)\mapsto \not{D}_{\mathbb{A}}(v)=\not{D}_{\mathbb{A}(0)}(v)+a.v$

$$\Omega^0(Y,W)\times\Omega^1(Y,i\mathbb{R})\xrightarrow{\not{D}_{\mathbb{A}}}\Omega^0(Y,W) \tag{51}$$

which is the derivative of a similarly defined map

$$\Omega^0(Y,W)\times\mathcal{A}(K)\to\Omega^0(Y,W) \tag{52}$$

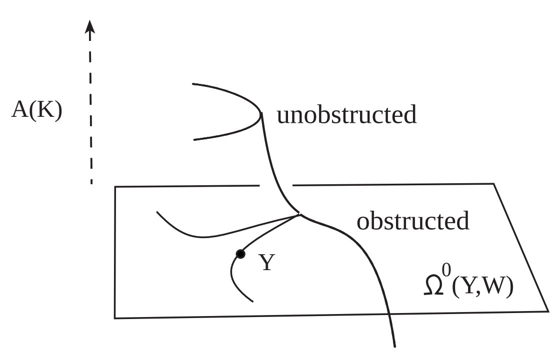

Figure 3.

In each slice $\mathbb{A}(a)$, we are deforming along normal vector fields by the connection $\mathbb{A}(a)$, which is a perturbation of the background connection $\mathbb{A}(0)$. To get compactness we can cut down this parametrized moduli space with an additional equation (induced from the map (50)) of the Seiberg-Witten theory $\Psi^{-1}(0)$, where

$$\Psi : \Omega^0(Y, W) \times \mathcal{A}(K) \to \Omega^0(Y, W) \times \Omega^2(Y, i\mathbb{R})$$

$$\begin{aligned} \not{D}_{\mathbb{A}}(v) &= 0 \\ *F_A &= \sigma(v, v) \end{aligned} \tag{53}$$

where F_A is the curvature of the connection $A = A_0 + a$ in K, and $*$ is the star operator on Y. Note that Y comes equipped with the natural submanifold metric. Now we proceed exactly as in the Seiberg-Witten theory of 3-manifolds (e.g. [C], [Lim], [Ma], [W]). To obtain smoothness of $\Psi^{-1}(0)$, we perturb the equations by 1-forms $\delta \in \Omega^1(Y)$ and get a new equation $\Phi = 0$, where

$$\Phi : \Omega^0(Y, W) \times \mathcal{A}(K) \times \Omega^1(Y) \to \Omega^0(Y, W) \times \Omega^1(Y, i\mathbb{R})$$

$$\begin{array}{rcl} \not{D}_{\mathbb{A}}(v) &=& 0 \\ *F_A + i\delta &=& \sigma(v, v) \end{array}$$

We can choose the perturbation term universally $\delta = f^*(\Delta)$, where $\Delta \in \Omega^1(M)$. Then Φ has a linearization:

$$D\Phi_{(v_0, A_0, 0)} : \Omega^0(Y, W) \times \Omega^1(Y, i\mathbb{R}) \times \Omega^1(Y) \to \Omega^0(Y, W) \times \Omega^1(Y, i\mathbb{R})$$

$$D\Phi_{(v_0, A_0, 0)}\ (v, a, \delta) = (\not{D}_{A_0}(v) + a.v_0,\ *da + i\delta - 2\sigma(v_0, v))$$

We see that $\Phi^{-1}(0)$ is smooth and the projection $\Phi^{-1}(0) \to \Omega^1(Y)$ is onto, so by Sard's theorem for a generic choice of δ we can make $\Phi_\delta^{-1}(0)$ smooth, where $\Phi_\delta(v, A) = \Phi(v, A, \delta)$. The bundle W of Y has a complex structure, so the gauge group $\mathcal{G}(K) = Map(Y, S^1)$ acts on the solution set $\Phi_\delta^{-1}(0)$, and makes the quotient $\Phi_\delta^{-1}(0)/\mathcal{G}(K)$ a smooth zero-dimensional manifold. This is because the infinitesimal action of $\mathcal{G}(K)$ on the complex $\Phi_\delta : \Omega^0(Y, W) \times \mathcal{A}(L) \to \Omega^0(Y, W) \times \Omega^1(Y, i\mathbb{R})$ is given by the map

$$\Omega^0(Y, i\mathbb{R}) \xrightarrow{G} \Omega^0(Y, W) \times \Omega^1(Y, i\mathbb{R})$$

where $G(f) = (fv_0, df)$. So after dividing by $\mathcal{G}$, tangentially the complex Φ_δ becomes

$$\Omega^0(Y; i\mathbb{R}) \xrightarrow{G} \Omega^0(Y, W) \times \Omega^1(Y, i\mathbb{R}) \to \Omega^0(Y, W) \times \Omega^1(Y, i\mathbb{R})/G$$

Hence the index of this complex is the sum of the indices of the Dirac operator $\not{D}_{\mathbb{A}_0} : \Omega^0(Y, W) \to \Omega^0(Y, W)$ (which is zero), and the index of the following complex

$$\Omega^0(Y, i\mathbb{R}) \times \Omega^1(Y, i\mathbb{R}) \to \Omega^0(Y, i\mathbb{R}) \times \Omega^1(Y; i\mathbb{R})$$

given by $(f, a) \mapsto (d^*(a), df + *da)$, which is also zero since Y^3 has zero Euler characteristic. Furthermore, $\Phi_\delta^{-1}(0)/\mathcal{G}(K)$ is compact and oriented (the same proof as in the Seiberg-Witten theory). Hence we get a number $SW_Y(M)$. Here we don't worry about metric dependence of $SW_Y(M)$ since we have a fixed background metric

induced from the G_2 structure. Hence we associated a number to a Λ-associative submanifold Y of (M, φ, Λ). In particular, Y moves in an unobstructed way along the parametrized sections the complex normal bundle $\Omega^0(Y, W) \times \mathcal{A}(L)$. Furthermore all these constructions work for almost Λ-associative submanifolds. So we have:

Theorem 8. *Let Y be an almost Λ-associative submanifold of (M, φ, Λ). By cutting down the space of parametrized complex deformations of Y with an additional equation as in (53) we obtain a zero dimensional compact smooth oriented manifold, hence we can associate a number $\Lambda_\varphi(Y) \in \mathbb{Z}$.*

Remark 6. *Clearly $\Lambda_\varphi(Y)$ is invariant under small isotopies through almost Λ-associative submanifolds $Y \subset (M, \varphi, \Lambda)$.*

The equations (53) can be induced universally from equations on (M^7, φ, Λ) by restriction: The 2-frame field $< u, v >$ gives a splitting of the tangent bundle $TM = \mathbf{E} \oplus \mathbf{V}$ with an $SO(3)$ bundle $\mathbf{E} =< u, v, u \times v >$ and a $U(2)$-bundle $\mathbf{V} = \mathbf{E}^\perp$, such that $\lambda_+(\mathbf{V}) = \mathbf{E}$. Let $\mathbf{W} \to M$ be the induced $\mathbb{C}^2$-bundle, and $\mathbf{K} \to M$ be the determinant line bundle of $\mathbf{W}$. We can define an action $T^*(M) \otimes \mathbf{W} \to \mathbf{W}$: For $w = x + y \in TM$, with $x \in \mathbf{E}$, $y \in \mathbf{V}$ and $z \in \mathbf{W}$ with $w.z = xz$. It is easy to check that this is a partial Clifford action, i.e. $w.(w.z) = -|x|^2 z$ Id, and it extends to an action $\Lambda^2(T^*M) \otimes \mathbf{W} \to \mathbf{W}$, and we have the map $\sigma : \mathbf{W} \otimes \mathbf{W} \to \mathbf{E}_{\mathbb{C}}$ of (50).

These bundles inherit connections from the Levi-Civita connection of (M, g_φ). Let $\mathcal{A}(\mathbf{K})$ be the connections on $\mathbf{K}$. Let A_0 denote the background connections. Then any $A \in \mathcal{A}(\mathbf{K})$ along with A_0 determines a connection on $\mathbf{W}$. Write $A = A_0 + a$ with $a \in \Omega^1(M)$. Hence for $A \in \mathcal{A}(\mathbf{K})$ we can define a partial Dirac operator $\not{D}_A(v) = \not{D}_{A_0}(v) + a.v$ on $\mathbf{W} \to M$, which is the composition:

$$\Omega^0(M, \mathbf{W}) \xrightarrow{\nabla_A} \Omega^0(M, T^*M \otimes \mathbf{W}) \xrightarrow{c_\varphi} \Omega^0(M, \mathbf{W})$$

We can now write the global version of the equations (54) on M in the usual way

$$\phi : \Omega^0(M, \mathbf{W}) \times \mathcal{A}(\mathbf{L}) \to \Omega^0(M, \mathbf{W}) \times \Omega^1(M) \quad \text{which is}$$

$$\begin{aligned} \not{D}_A(v) &= 0 \\ *F_A &= \sigma(v, v) \end{aligned} \tag{54}$$

where $* : TM \to TM$ is the star operator on $\mathbf{E}$ and zero on $\mathbf{V}$. We can perturb these equations by 1-forms to $\Phi = 0$, and proceed as before. $\mathbf{W}$ has a complex structure. The gauge group $\mathcal{G}(\mathbf{L}) = Map(M, S^1)$ acts on the solution set $\Phi^{-1}(0)$, and the quotient $\Phi^{-1}(0)/\mathcal{G}(\mathbf{L})$ can be formed. To sum up we have:

Proposition 9. *Any almost Λ-associative submanifold $f : Y^3 \hookrightarrow (M, \varphi, \Lambda)$ pulls back the equations (54) to the Seiberg-Witten equations (53) on Y.*

8. Associative 3-Plane Fields of G_2 Manifolds

Recall that, any non-vanishing oriented 2-plane field $\Lambda =< u, v >$ on (M, φ) determines a section $\Lambda_\varphi : M \to \tilde{M}_\varphi \subset \tilde{M}$. In particular, it gives a non-vanishing associative 3-plane field $\mathbf{E} = \mathbf{E}_{\Lambda,\varphi} \to M$ on M, and a complex structure on the complementary 4-plane field $\mathbf{V} = \mathbf{V}_{\Lambda,\varphi} \to M$, and a splitting $TM = \mathbf{E} \oplus \mathbf{V}$, with $\lambda_+(\mathbf{V}) = \mathbf{E}$. From the construction we get a further splitting $\mathbf{E} = \mathbf{\Lambda} \oplus \xi$, corresponding to $< u, v > \oplus < u \times v >$. The orientation of the 2-dimensional bundle $\mathbf{\Lambda}$ gives it a complex structure, and we have

$$TM = \bar{\mathbf{E}} \oplus \xi \tag{55}$$

where $\bar{\mathbf{E}} = \mathbf{\Lambda} \oplus \mathbf{V}$ is a 6-plane bundle with a complex structure and ξ is the line bundle $< u \times v >$. Note that if φ is integrable and the vector field $u \times v$ is parallel then M would be a Calabi-Yau $\times S^1$ (since G_2 holonomy would reduce to $SU(3)$). So non-vanishing oriented 2-plane fields may be thought of objects taming the G_2 structure. Any integral submanifold of the corresponding distribution $\mathbf{E}$ is an associative submanifold $Y^3 \subset M$ with a $Spin^c$-structure (i.e. the 2-plane field $\xi = \Lambda|_Y$).

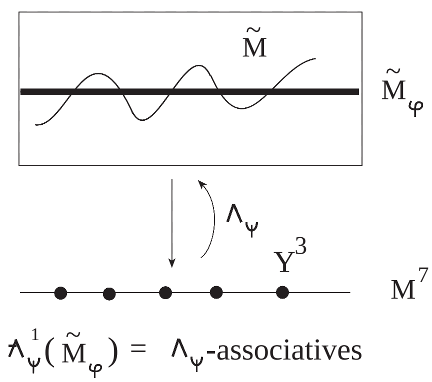

Figure 4.

By fixing the plane field Λ, and varying $\varphi \in \tilde{\Omega}^3_+(M)$ (the set of G_2 structures inducing the same metric on M) has the effect of varying $\xi \in \Lambda^\perp$ (the cross product operation on Λ) and varying the complex structure on $\mathbf{V} = (\Lambda \oplus \xi)^\perp$. These ξ's are the sections of the S^4-sphere bundle of $\Lambda^\perp \to M^7$, hence generically any other section will agree with Λ_φ on some 3-manifold $Y \subset M$. We will show that this 3-manifold is almost Λ-associative. First consider the parametrized section:

$$\mathbf{\Lambda} : \tilde{\Omega}^3_+(M) \times M \to \tilde{M} \tag{56}$$

$(\lambda, x) \mapsto \Lambda_\lambda(x)$. By Lemma 5 there is an identification $\tilde{\Omega}^3_+(M) = \{s^*(\varphi) \mid s \in \mathcal{G}(P)\}$ (the sections of an $\mathbb{RP}^7$ bundle over M). We claim $\mathbf{\Lambda}$ is transversal to $\tilde{M}_\varphi$.

First we need to recall a few facts: By [B2], the deformations of the G_2 structure φ fixing the metric $g = g_{\varphi}$, are parametrized by φ_{λ} below, where $\lambda = [a, \alpha]$ are the sections $\tilde{\Omega}^3_+(M)$ of the $\mathbb{RP}^7$-bundle, which is the projectivization $P(\mathbb{R} \oplus T^*M) \to M$

$$\varphi_{\lambda} = (a^2 - |\alpha|^2)\varphi + 2a * (\alpha \wedge \varphi) + 2\alpha \wedge *(\alpha \wedge *\varphi)$$

where $a^2 + |\alpha|^2 = 1$. From the identities $*(\alpha \wedge \varphi) = -\alpha^{\#} \lrcorner * \varphi$ and $*(\alpha \wedge * \varphi) = \alpha^{\#} \lrcorner \varphi$, where $\alpha^{\#}$ is the metric dual of α, we can also express

$$\varphi_{\lambda} = \varphi - 2\alpha^{\#} \lrcorner \, [\, a(*\varphi) + \alpha \wedge \varphi \,] \tag{57}$$

$$*\varphi_{\lambda} = *\varphi + 2\alpha \wedge [\, a\varphi - (\alpha^{\#} \lrcorner \, * \varphi) \,]$$

Not to clutter notations, we denote $\Lambda_{\lambda} = \Lambda_{\varphi_{\lambda}}$ and use the metric to identify $T^*(M) = \mathbf{E} \oplus \mathbf{V}$, and identify M with the zero section of the bundle $\mathbf{V} \to M$.

Theorem 10. *For $\alpha \in \Omega^1(M)$ which is a transverse section of $\mathbf{V} \to M$, the map Λ_{λ}, where $\lambda = [a, \alpha]$ and $a \neq 0$, is transversal to $\tilde{M}_{\varphi}$, and $\Lambda_{\lambda}^{-1}(\tilde{M}_{\varphi}) = \alpha^{-1}(M)$.*

Proof. The set $\Lambda_{\lambda}^{-1}(\tilde{M}_{\varphi})$ is given by the solutions of the equation $\Lambda_{\varphi_{\lambda}}(x) = \Lambda_{\varphi}(x)$, where $\varphi \mapsto \varphi_{\lambda}$ is a deformation of φ. Since $\mathbf{E}$ is obtained from the oriented 2-plane field $\Lambda = < u, v >$ by association $< u, v > \mapsto < u, v, u \times_{\varphi} v >$, this equation is equivalent to $(u \times v)_{\lambda}(x) = (u \times v)(x)$ (up to positive scalar multiple), where $(u \times v)_{\lambda}$ denotes the cross product corresponding to φ_{λ}. By using $(u \times v)^{\#} = u \lrcorner \, v \lrcorner \, \varphi$, we can calculate the deviation of the cross product operation under the deformation

$$\begin{aligned} (u \times v)_{\lambda} &= (1 - 2|\alpha|^2)(u \times v) + 2\, [\, - a\chi(u, v, \alpha^{\#}) \\ &\quad + \alpha(v)(u \times \alpha^{\#}) - \alpha(u)(v \times \alpha^{\#}) + \varphi(u, v, \alpha^{\#})\alpha^{\#} \,] \end{aligned}$$

So the equation $(u \times v)_{\lambda}(x) = (u \times v)(x)$ is given by the equation $F = 0$ where:

$$F = a\chi(u, v, \alpha^{\#}) - \alpha(v)(u \times \alpha^{\#}) + \alpha(u)(v \times \alpha^{\#}) - \varphi(u, v, \alpha^{\#})\alpha^{\#} \; + |\alpha|^2(u \times v)$$

Note that when $\alpha^{\#} \in \mathbf{E}$, the equation $F(x) = 0$ holds for all $x \in M$. Let us choose our deformation $\alpha^{\#} \in \mathbf{V}$, which is a transverse section of $\mathbf{V} \to M$. In this case by (4), Lemma 1, and Lemma 2 the equation $F(x) = 0$ is equivalent to

$$aJ(\alpha^{\#}) = -|\alpha|^2(u \times v)$$

where J is the complex structure defined in Lemma 2. Since $J(\alpha^{\#}) \in \mathbf{V}$ and $u \times v \in \mathbf{E}$, this equation holds only at points satisfying $\alpha^{\#}(x) = 0$. By taking derivative of $F(a, \alpha)$ we see that F is transversal to $\tilde{M}_{\varphi}$ when $a \neq 0$. ∎

9. Cayley Submanifolds of $Spin(7)$

Much of what we have discussed for associative submanifolds of a G_2 manifold holds for Cayley submanifolds of a $Spin(7)$ manifold. Let (N^8, Ψ) be a $Spin(7)$ manifold, and $\mathcal{P}_{Spin(7)}(N) \to N$ be its $Spin(7)$ frame bundle, and $G(4,8)$ be the Grassmannian of oriented 4 planes in $\mathbb{R}^8$. As in the G_2 case we can form the bundle

$$\tilde{N} = \mathcal{P}(N) \times_{SO(8)} G(4,8) \to N.$$

Similarly we have the universal bundles Ξ , $\mathbb{V} \to \tilde{N}$ which are fiberwise extensions of the canonical bundle $\xi \to G(4,8)$ and its dual $\nu = \xi^{\perp} \to G(4,8)$, respectively. $Hom(\Xi, \mathbb{V}) = \Xi^* \otimes \mathbb{V} \to \tilde{N}$ is the vertical subbundle of $T(\tilde{N}) \to N$ with fibers $TG(4,8)$. Let $G^{\Psi}(4,8)$ be the Grassmannian of Cayley 4-planes in $G(4,8)$ consisting of elements $L \in G(4,8)$ satisfying $\Psi|L = vol(L)$. The group $Spin(7)$ acts transitively on $G^{\Psi}(4,8)$ with the stabilizer $(SU(2) \times SU(2) \times SU(2))/\mathbb{Z}_2$. Therefore, $G^{\Psi}(4,8)$ can be identified by the quotient of $Spin(7)$ with the subgroup

$$(SU(2) \times SU(2) \times SU(2))/\mathbb{Z}_2 \subset Spin(7).$$

The action of $[q_+, q_-, \lambda] \in (SU(2) \times SU(2) \times SU(2))/\mathbb{Z}_2$ on $\mathbb{R}^8 = \mathbb{H} \oplus \mathbb{H}$ is given by $(x, y) \to (q_+ x q_-^{-1}, q_+ y \lambda^{-1})$. As in G_2 case there is the Cayley Grassmannian bundle

$$\tilde{N}_{\Psi} = \mathcal{P}_{Spin(7)}(N) \times_{Spin(7)} G^{\varphi}(4,8) \to N$$

which is $\tilde{N}_{\Psi} = \mathcal{P}(N)/(SU(2) \times SU(2) \times SU(2))/\mathbb{Z}_2 \to \mathcal{P}(N)/Spin(7) = N$. We have restriction of the bundles Ξ^*, $\mathbb{V} \to \tilde{N}_{\Psi} \subset \tilde{N}$. Furthermore, the principal $(SU(2) \times SU(2) \times SU(2))/\mathbb{Z}_2$ bundle $\mathcal{P}(N) \to \tilde{N}_{\Psi}$ gives the following associated vector bundles over $\tilde{N}_{\Psi}$ via the representations (see [HL], [M]).

$$\begin{array}{ll} \mathbb{W}^+ = \mathbb{V}: & y \mapsto q_+ y \lambda^{-1} \\ \mathbb{W}^-: & y \mapsto q_- y \lambda^{-1} \\ \Xi^*: & x \mapsto q_+ x q_-^{-1} \\ \lambda_+(\Xi^*): & x \mapsto q_+ x q_+^{-1} \\ \lambda_-(\Xi^*): & x \mapsto q_- x q_-^{-1} \\ \lambda_-(\mathbb{W}^+): & x \mapsto \lambda x \lambda^{-1} \end{array} \tag{58}$$

where $[q_+, q_-, \lambda] \in SU(2) \times SU(2) \times SU(2))/\mathbb{Z}_2$. We can identify: $\lambda_+(\mathbb{W}^+) = \lambda_+(\Xi^*)$, and we have the usual decomposition $\Lambda^2(\Xi^*) = \lambda_+(\Xi^*) \oplus \lambda_-(\Xi^*)$. We have the Clifford multiplications $\Xi^* \otimes \mathbb{W}^{\pm} \to \mathbb{W}^{\mp}$ given by: $x \otimes y \mapsto -\bar{x}y$ and $x \otimes y \mapsto xy$, on $\mathbb{W}^+$ and $\mathbb{W}^-$ respectively, which extends to $\Lambda^2(\Xi^*) \otimes \mathbb{W}^+ \to W^+$.

The Gauss map of an imbedding $f : X^4 \hookrightarrow N^8$ of any 4-manifold canonically lifts to an imbedding $\tilde{f} : X^4 \hookrightarrow \tilde{N}$, and the pull backs $\tilde{f}^*\Xi^* = T^*(X)$ and $\tilde{f}^*\mathbb{W}^+ = \nu(X)$ give cotangent and normal bundles of X. Furthermore, if X is a Cayley submanifold of N then the image of $\tilde{f}$ lands in $\tilde{N}_{\Psi}$; in this case pulling back the principal $Spin(7)$ frame bundle $\mathcal{P}(N) \to N$ induces an $(SU(2) \times SU(2) \times SU(2))/\mathbb{Z}_2$ bundle $\mathcal{P}(X) \to X$. So by the representations (58) we get associated vector bundles

$W^+ = \nu(X), W^-, T^*(X)$ over X, i.e. the pull-backs of $\mathbb{W}^+, \mathbb{W}^-, \Xi^*$. So we have the actions $W^+ \otimes \lambda_-(W^+) \to W^+$ and $T^*X \otimes W^\pm \to W^\mp$ and $\Lambda^2(T^*X) \otimes W^+ \to W^+$.

The Levi-Civita connection induced by the $Spin(7)$ metric on N, induces connections on tangent and normal bundle of any submanifold $X^4 \subset N$. Call these connections background connections. Let $\mathbb{A}_0$ be the induced connection on $\nu(X) = W^+$. Using the Lie algebra decomposition $so(4) = so(3) \oplus so(3)$, we can decompose $\mathbb{A}_0 = S_0 \oplus A_0$, where S_0 and A_0 are connections on $\lambda_+(T^*X)$ and $\lambda_-(W^+)$, respectively. Any connection A of $\lambda_-(W^+)$ is in the form $A = A_0 + a$ where $a \in \Omega^1(X, \lambda_-(W^+))$, and by the association $A \mapsto S_0 \oplus A$ it induces a connection on $\nu(X)$. We will denote this connection by $\mathbb{A} = \mathbb{A}(a)$, and $\mathbb{A}_0 = \mathbb{A}(0)$. Later we will consider deformations

$$\mathbb{A}_0 \mapsto \mathbb{A}. \tag{59}$$

Let $\nabla_{\mathbb{A}} : \Omega^0(X, W^+) \to \Omega^1(X, W^+)$ by $\nabla_A = \sum e^i \otimes \nabla_{e_i}$, where $\{e_i\}$and $\{e^i\}$ are orthonormal tangent and cotangent frame fields of X, respectively. When X is a Cayley manifold, the Clifford multiplication gives the twisted Dirac operator:

$$\not{D}_{\mathbb{A}} : \Omega^0(X, W^+) \to \Omega^0(X, W^-) \tag{60}$$

The kernel of $\not{D}_{\mathbb{A}_0}$ gives the infinitesimal deformations of Cayley submanifolds ([M]). As in the associative case by deforming $\mathbb{A}_0 \to \mathbb{A}$ we can make cokernel of $\not{D}_{\mathbb{A}}$ zero.

Similar to the case of Λ-associative submanifolds in G_2 manifolds, we can study the Cayley submanifolds in $Spin(7)$ manifolds with complex normal bundles. There are several ways of lifting various subbundles to complex bundles, for example

$$Spin^c(4) = (SU(2) \times SU(2) \times S^1)/\mathbb{Z}_2 \subset (SU(2) \times SU(2) \times SU(2))/\mathbb{Z}_2$$

gives a $Spin^c(4)$ bundle $\mathcal{P}(N) \to \bar{N}_\Psi = \mathcal{P}(N)/Spin^c(4)$, and we have all the corresponding bundles of (58) over $\bar{N}_\Psi$ (except in this case we have $\lambda \in S^1$). The S^2-bundle $\bar{N}_\Psi \to \tilde{N}_\Psi$ can be identified with the sphere bundle of $\lambda_-(\mathbb{W}^+) \to \tilde{N}_\Psi$, and the sections of this bundle correspond to almost complex structures on $\mathbb{W}^\pm$. Previously, in the case of 7-manifolds, existence of such sections followed from the existence of 2-frame field [T], in the 8-dimensional $Spin(7)$ case we don't have a clean analogue of [T], so in this case we will make this an assumption and proceed. So consider a $Spin(7)$ manifold (N^8, Ψ, Λ) with a unit section $\Lambda : \tilde{N}_\Psi \to \lambda_-(\mathbb{W}^+)$. Hence $\mathbb{W}^\pm \to \bar{N}_\Psi$ are $U(2)$ bundles, and $\lambda_-(W^+)$ is a line bundle $L \to \bar{N}_\Psi$. As in (50) there is a quadratic bundle map $\sigma : \mathbb{W}^+ \otimes \mathbb{W}^+ \to \lambda_+(\Xi^*)$

$$\sigma(x, x) = -\frac{1}{2}(xi\bar{x})i.$$

Now if $f : X^4 \hookrightarrow N^8$ is a Cayley submanifold, we can pull back these structures onto X by $\Lambda \circ \tilde{f}$. Then we can "perturb" the local Cayley deformations of X by deforming the connection as in (59), i.e. the kernel of the Dirac operator of (60).

Then if we can cut down the solution space $\not{D}_{\mathbb{A}}^{-1}(0)$ by a second natural equation (by using "a" as a free wariable) we arrive to the Seiberg-Witten equations:

$$\begin{array}{rcl} \not{D}_A(v) & = & 0 \\ F_A^+ & = & \sigma(v,v) \end{array} \tag{61}$$

As usual, by perturbing these equations by elements of $\Omega^2_+(X)$, i.e. by changing the second equation with $F_A^+ + \delta = \sigma(v,v)$ with $\delta \in \Omega^2_+(X)$ we get smoothness on the zero locus of the parameterized equation $F = 0$ where

$$F : \Omega^0(X, W^+) \times \mathcal{A}(L) \times \Omega^2_+(X) \to \Omega^0(X, W^-) \times \Omega^2_+(X)$$

and by generic choice of δ we can make the solution set $F_\delta^{-1}(0)$ smooth. The normal bundle W^+ of X has a complex structure, so the gauge group $\mathcal{G}(L) = Map(X, S^1)$ acts on $F_\delta^{-1}(0)$, and makes quotient $F_\delta^{-1}(0)/\mathcal{G}(L)$ a smooth manifold whose dimension d can be calculated from the index of the elliptic complex:

$$\Omega^0(X) \to \Omega^0(X, W^+) \times \Omega^1(X) \to \Omega^0(X, W^-) \times \Omega^2_+(X) \tag{62}$$

where the first map comes from gauge group action. As in Seiberg-Witten we get:

$$d = \frac{1}{4}\left[c_1^2(L) - (2e(X) + 3\sigma(X)\right] \tag{63}$$

Here e and σ denote the Euler characteristic and the signature. In particular, these parametrized deformations of complex Cayley submanifolds in $\Omega^0(X, W^+) \times \mathcal{A}(L)$ are unobstructed.

Theorem 11. *Given (X, Ψ, Λ), to any Cayley submanifold $f : X^4 \hookrightarrow N$ we can assign a number $\Lambda_\Psi(X) \in \mathbb{Z}$. Furthermore, the Seiberg-Witten equations of (61) can be pulled back by f from global equations on N (analogue of Proposition 9).*

Note that $SU(3)$ and G_2 also act on the corresponding special Lagrangian and coassociative Grassmannians with $SO(3)$ and $SO(4)$ stabilizers, respectively [HL], giving the identifications $G^{SL}(3,6) = SU(3)/SO(3)$ and $G^{coas}(4,7) = G_2/SO(4)$. As before, one can study special Lagrangians in a Calabi-Yau manifold, and coassociative submanifolds in a G_2 manifold, by lifting their normal bundles to $SU(2)$. Their deformation spaces are unobstructed and can be identified with H^1 and H^2_+, respectively. With a similar approach we can relate them to the reduced Donaldson invariants (as the Λ-associative and similarly defined Cayley's are related to Seiberg-Witten invariants). Similarly one can treat the deformations of associative submanifolds whose boundaries lie on coassociative submanifolds, and the Cayley's in $Spin(7)$ with associative boundaries in G_2. Also asymptotically cylindrical associative submanifolds in a G_2 manifold with a Calabi-Yau boundary have similar local deformation spaces, their deformations are related to the corresponding holomorphic curves inside the Calabi-Yau boundary.

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